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Theory of Signals

On a generalized X-ray transform and a new method for defect detection using the medium electronic density

Sur une transformation de type rayons X généralisée et une nouvelle méthode de détection de défauts utilisant la densité électronique en contrôle non-destructif

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Abstract

We consider a generalization of the X-ray transform which maps a function $f(\mathbf{V})$ defined in $\mathcal{R}^3$ onto its integrals over a pair of parallel straight lines and not over a single straight line as in a conventional X-ray transform. This new transformation arises from the image formation by double Compton-scattered radiation in transmission imaging. The problem of reconstructing $f(\mathbf{V})$ from its line pair integrals is formulated as an inverse problem for this generalized X-ray transform. Exploiting the line duality in the new transformation, we derive an equivalent nonlinear integral equation for $f(\mathbf{V})$. Special classes of solutions can be constructed. They may serve as basis for a new method of defect detection, using the medium electronic density, in non-destructive inspection. ***To cite this article: M.K. Nguyen, T.T. Truong, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On considère une généralisation de la transformation de type rayons X, qui associe à une fonction $f(\mathbf{V})$ définie dans $\mathcal{R}^3$ ses intégrales sur une paire de lignes droites parallèles, au lieu d'une ligne droite comme dans le cas de la transformation conventionnelle de type rayons X. Elle provient de la formation d'image par le rayonnement doublement diffusé par effet Compton en imagerie par transmission. La reconstruction de $f(\mathbf{V})$ est présentée comme un problème inverse de cette transformation généralisée de type rayons X. Exploitant la dualité entre les lignes droites parallèles dans la nouvelle transformation, on montre de façon équivalente que $f(\mathbf{V})$ vérifie une équation intégrale non-linéaire, pour laquelle des solutions spéciales peuvent être construites. Ces solutions peuvent servir de base à une nouvelle méthode de détection de défauts par le

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biais de la densité électronique du milieu en contrôle non-destructif. ***Pour citer cet article : M.K. Nguyen, T.T. Truong, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

L'analyse des images formées par la double diffusion Compton, due aux électrons libres, d'un pinceau de rayons X ou gamma incident sur un bloc de matériau conduit à une relation intégrale du genre transformation de type rayons X. Cependant dans cette transformation, la fonction inconnue, en l'occurence la densité électronique n_e du milieu, intervient sous la forme d'un produit bilocal $n_e(\mathbf{M})n_e(\mathbf{N})$, où les sites **M** et **N** se trouvent sur deux droites parallèles. On appellera cette nouvelle transformation *la transformation généralisée de type rayons X*. On exploite la dualité entre ces droites dans la nouvelle transformation pour déduire de toute série d'images une autre série d'images dite série permutée. L'ensemble de ces images forme les données suffisantes à la reconstruction de la densité électronique volumique (problème inverse). Ceci permet de remplacer les relations intégrales bilocales en n_e par une équation intégrale non-linéaire dans laquelle la fonction inconnue est locale. A notre connaissance, la solution exacte de cette équation n'est pas connue [1]. Cependant on peut construire des classes de solutions spéciales par découplage ou par linéarisation. Ce résultat théorique peut servir de base à un nouveau principe de détection des défauts par le biais de n_e en contrôle non-destructif. Etant donné que les images paramétrées par l'angle de diffusion θ sont obtenues sans avoir recours au mouvement relatif du système source-détecteur la méthode utilisant le rayonnement diffusé présente un avantage technique considérable. De plus, bien que la nouvelle détection de défauts via n_e – densité électronique, soit complémentaire de la détection conventionnelle via μ – le coefficient d'atténuation linéaire, ces deux modalités d'imagerie peuvent être réalisées sur le même appareillage fonctionnant soit avec le rayonnement diffusé soit avec le rayonnement primaire (non diffusé).

1. Introduction

Integral transforms form the mathematical core of most imaging methods by ionizing radiation. Let $f(\mathbf{M})$ be a function defined in $\mathcal{R}^3$ describing a physical quantity of the object under study and $g(\mathsf{D},\tau)$ be the data measured by the imaging apparatus, such as a planar detector, with $\mathsf{D} \in \mathcal{R}^2$ and $\tau \in \mathcal{R}$, a relevant parameter. The image formation process implies a mapping $\mathcal{T}$ such that $\mathcal{T} : f(\mathbf{M}) \mapsto g(\mathsf{D},\tau)$. Finding $\mathcal{T}^{-1}$ or giving the conditions and process under which $f(\mathbf{M})$ can be reconstructed from $g(\mathsf{D},\tau)$ constitutes the inverse problem to solve.

In modern Computer Assisted Tomography (CAT), the integral mapping of interest is the so-called X-ray transform. The unknown function $f(\mathbf{M})$ is $\mu(\mathbf{M})$, the linear attenuation coefficient of X-ray in the traversed medium, and the measured data $g(\mathsf{D},\tau)$ is the integral of $f(\mathbf{M})$ along a straight line $\mathcal{L}$ joining the source point **V** to the detection point D (see Fig. 1):

$$g(\mathsf{D},\tau) = \int_{\mathbf{M}\in\mathcal{L}(\mathsf{D},\tau)} \mathrm{d}s\, f\big(\mathbf{M}(s)\big), \tag{1}$$

where τ is a parameter representing the line $\mathcal{L}$.

The idea of exploiting Compton scattered radiation for imaging goal has been proposed for the first time in Compton scatter tomography [5]. However, quite recently, it has been shown that, in emission imaging, *single* Compton scattered events do generate an innovative imaging process exhibiting some advantages over conventional ones [2–4]. First the scattered radiation generates a new class of so-called secondary images, expressed as conical Radon transform of the object density, for which the existence of an exact inverse is proved. Then accurate reconstruction of the object can be achieved. Second in this new imaging modality a sufficient set of data (series of images parameterized by the scattering angle) can be obtained without requiring the motion of the detector.

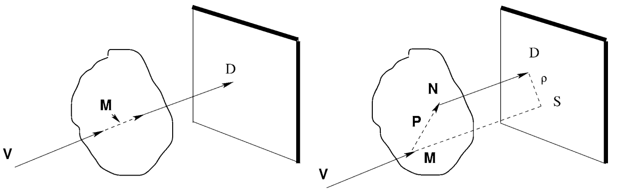

Fig. 1. Working principle of X-ray transform and generalized X-ray transform.

Fig. 1. Principe de la transformation, et de la tranformation généralisée de type rayons X.

In this Note, we go one step further and show that *double* Compton scattered radiation is also able to produce a new imaging method, which is based on a transform of the X-ray type but *quadratic* and *bilocal* in the unknown function $f(\mathbf{M})$:

$$g(\mathsf{D}, \mathbf{P}) = \int_{\mathbf{M}\in\mathcal{L}(\mathsf{D},\mathbf{P})} \mathrm{d}s\, f\big(\mathbf{M}(s)\big) f\big(\mathbf{M}(s) + \mathbf{P}\big), \tag{2}$$

where $\mathbf{P}$ is a vector joining two scatter sites $\mathbf{M}$ and $\mathbf{N}$ and plays the role of τ.

It is clear that Eq. (2) is a natural generalization of the standard X-ray transform in which the product of the same function at two different points in space is summed up along a pair straight lines. The question is now how to reconstruct $f(\mathbf{M})$ from the data $g(\mathsf{D}, \mathbf{P})$.

In the present paper, we show how image formation by double-Compton scattered radiation leads to this *generalized X-ray transform*. Then taking advantage of the duality between the pair of straight lines in this transform, we establish nonlinear integral equations in which unknown functions are local instead of bi-local as in the original integral relations. Two classes of special solutions are constructed and discussed. They turn out to be appropriate for establishing a new method of defect detection, via the medium electronic density in non-destructive inspection.

2. The transmission imaging process by double scattered radiation

To concentrate only on the analysis of Compton scattering effect, we neglect all other radiation attenuations and use the Cartesian coordinate system of Fig. 2.

Consider now an ideal pencil beam of *unit* gamma or X-photon flux density irradiating a piece of bulk material. It may encounter $n_e(\mathbf{M})$ electrons at site $\mathbf{M}$ and get scattered off into a solid angle $\mathrm{d}\Omega_N$ defined by a deflection angle θ. Such a deflected ray is “lost” for the collimated detector in transmission imaging, unless it re-scatters at site $\mathbf{N}$ with the same angle θ. It reaches the detector at site D, within the solid angle $\mathrm{d}\Omega_D$ after encountering $n_e(\mathbf{N})$ electrons. The angle θ is here related to the outgoing radiation energy $E(\theta)$ by the Compton formula:

$$E(\theta) = E_0\big[1 + 2\varepsilon(1 - \cos\theta)\big]^{-1}, \tag{3}$$

where $\varepsilon = E_0/mc^2$, E_0 and mc^2 are respectively the incident radiation energy and the electron rest energy. The factor 2 is due to the geometry of parallel lines.

For the geometry of Fig. 2, the photon flux density at site D after scattering at sites $\mathbf{M}$ and $\mathbf{N}$ is:

$$h_{\mathsf{S}}(\mathsf{D}|\tau) = \int_{l}^{L-\rho\tau} \mathrm{d}\zeta_N\, \frac{K(\tau)}{\rho^2 {\zeta_N}^2}\, n_e(\mathbf{N}) n_e(\mathbf{M}), \tag{4}$$

where $\mathbf{N} = (\mathsf{D}, \zeta_N)$, resp. $\mathbf{M} = (\mathsf{S}, \zeta_M)$ with $(l < \zeta_N < L - \rho\tau)$, resp. $(l + \rho\tau < \zeta_M < L)$, $\tau = \cot\theta$, $\rho = |\mathsf{DS}|$ and $K(\tau)/\rho^2 {\zeta_N}^2$, a factor due to the Compton kinematics and spherical wave propagation at scattering sites.

Thus $h_{\mathsf{S}}(\mathsf{D}|\tau)$ is a generalized X-ray transform of the electron density n_e. It collects and integrates information along two parallel lines with a separation dependent on the scattering angle θ. $h_{\mathsf{S}}(\mathsf{D}|\tau)$ represents also the image of

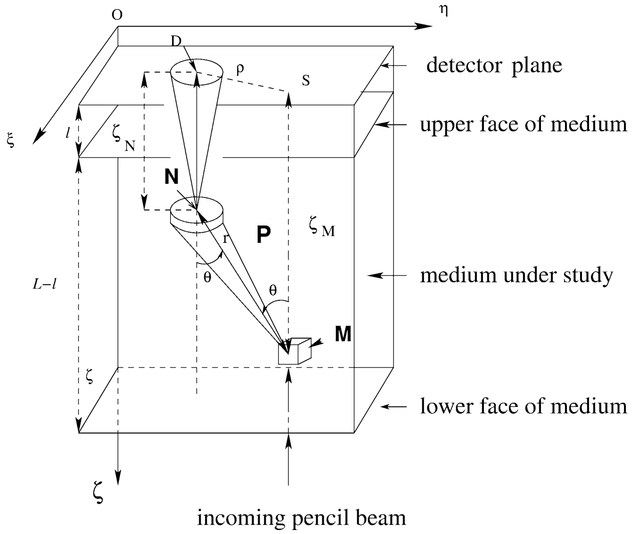

Fig. 2. Schematic representation of the functioning principle.

Fig. 2. Schéma du principe de fonctionnement.

an ideal pencil beam perpendicular to the detector at site S and will be called *Transmission Pencil Source Function* or TPSF. For a parallel incoming photon beam of finite cross section and of transverse flux density $\Phi_0(\mathsf{S})$, the total radiation flux density at a detector site D and scattering angle θ can be expressed with the use of the TPSF as

$$g(\mathsf{D}|\tau) = \int \mathrm{d}\mathsf{S}\, h_{\mathsf{S}}(\mathsf{D}|\tau)\, \Phi_0(\mathsf{S}). \tag{5}$$

3. Alternative formulation with a nonlinear integral equation

As functions of ζ, $n_e(\mathsf{D}, \zeta)$ and $n_e(\mathsf{S}, \zeta)$ are two different functions related by Eq. (4). To compute each of them, an additional relation is needed. Such a relation can be generated by a measurement process which interchanges the roles of D and S at fixed distance ρ. This doubling of the data is now encoded in the pair of TPSF expressions:

$$h_{\mathsf{S}}(\mathsf{D}|\tau) = C \int_l^{l'} \frac{\mathrm{d}\zeta_N}{\zeta_N^2} n_e(\mathsf{D}, \zeta_N) n_e(\mathsf{S}, \zeta_N + \rho\tau) \quad \text{and} \quad h_{\mathsf{D}}(\mathsf{S}|\tau) = C \int_l^{l'} \frac{\mathrm{d}\zeta_M}{\zeta_M^2} n_e(\mathsf{D}, \zeta_M + \rho\tau) n_e(\mathsf{S}, \zeta_M) \tag{6}$$

with $l' = L - \rho\tau$ and $C = K(\tau)/\rho^2$. This is a coupled system of equations bi-local in $n_e(\mathsf{D}, \zeta)$ and $n_e(\mathsf{S}, \zeta)$. To compute each of these functions, we must eliminate one of them from Eqs. (6) and obtain a single local equation in n_e. To this end let us define, instead of n_e a new unknown function:

$$N_e(\mathsf{R}, w) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta^2} Y(\zeta - l) Y(L - \rho\tau - \zeta) n_e(\mathsf{R}, \zeta)\, \mathrm{e}^{2\mathrm{i}\pi w \zeta}, \tag{7}$$

where R is one of the sites D or S on the detector (see Fig. 1) and $Y(\cdot)$ is the Heaviside function.

The set of TPSF $h_{\mathsf{S}}(\mathsf{D}|\tau)$ for various $\tau \in \mathcal{R}$ may be regarded as a set of images parameterized by τ. To solve the inverse problem, it is necessary to consider the *totality* of the images taken under the form of normalized data function:

$$Q_{\mathsf{S}}(\mathsf{D}|w) = \int_{-\infty}^{\infty} \mathrm{d}\tau \, \frac{h_{\mathsf{S}}(\mathsf{D}|\tau)}{K(\tau)} \, \mathrm{e}^{-2\mathrm{i}\pi w\tau}. \tag{8}$$

Eqs. (6) becomes now a pair of coupled nonlinear integral equations for $N_e(\mathsf{S}, w')$ and $N_e(\mathsf{D}, w'')$:

$$\left(\frac{N_e(\mathsf{S}, w')}{\rho^2}, \text{ resp. } \frac{N_e(\mathsf{D}, w')}{\rho^2}\right) = \int_{-\infty}^{\infty} \mathrm{d}u \, J_l(u + w')\big(Q_{\mathsf{D}}(\mathsf{S}|u\rho)\, N_e^{-1}(\mathsf{D}, u), \text{ resp. } Q_{\mathsf{S}}(\mathsf{D}|u\rho) N_e^{-1}(\mathsf{S}, u)\big), \tag{9}$$

with kernel:

$$J_l^L(w) = \int_{-\infty}^{\infty} \mathrm{d}s \, \frac{Y(s)Y(L - l - \rho\tau - s)}{(s + l)^2} \, \mathrm{e}^{-2\mathrm{i}\pi w(l+s)}. \tag{10}$$

Rewriting Eq. (9) with the compact notation of integral transforms $\mathcal{T}_{\mathsf{S},\mathsf{D}}[\cdot]$ and $\mathcal{T}_{\mathsf{D},\mathsf{S}}[\cdot]$ yields:

$$N_e(\mathsf{S}) = \mathcal{T}_{\mathsf{S},\mathsf{D}}\big[N_e(\mathsf{D})\big] \quad \text{and} \quad N_e(\mathsf{D}) = \mathcal{T}_{\mathsf{D},\mathsf{S}}\big[N_e(\mathsf{S})\big]. \tag{11}$$

We have thus the proposition:

Proposition 3.1. *The generalized X-ray transform as given by Eqs.* (6) *may be described alternatively by the set of nonlinear integral equations*:

$$N_e(\mathsf{S}, u) = \mathcal{T}_{\mathsf{S},\mathsf{D}}\big[\mathcal{T}_{\mathsf{D},\mathsf{S}}\big[N_e(\mathsf{S}, u)\big]\big] \quad \textit{and} \quad N_e(\mathsf{D}, u') = \mathcal{T}_{\mathsf{D},\mathsf{S}}\big[\mathcal{T}_{\mathsf{S},\mathsf{D}}\big[N_e(\mathsf{S}, u')\big]\big], \tag{12}$$

the electron density is recovered via the inverse Fourier transform of Eq. (7).

To our knowledge exact solutions of this nonlinear equation are not known. However accurate numerical solutions can be found [1]. In the following, we discuss special solutions obtained by “linearization” and “decoupling”.

4. A simple class of solutions by “linearization”

For small ρ with respect to any other relevant lengths but with $\rho \neq 0$, we can consider the approximate equations:

$$h_{\mathsf{S}}(\mathsf{D}|\tau) = \frac{K(\tau)}{\rho^2} \int_{l}^{l'} \frac{\mathrm{d}\zeta_N}{\zeta_N^2} n_e^2(\mathsf{D}, \zeta_N + \rho\tau) \quad \text{and} \quad h_{\mathsf{D}}(\mathsf{S}|\tau) = \frac{K(\tau)}{\rho^2} \int_{l}^{l'} \frac{\mathrm{d}\zeta_M}{\zeta_M^2} n_e^2(\mathsf{S}, \zeta_M + \rho\tau). \tag{13}$$

These are integral equations *linear*, in n_e^2. They may be viewed as X-ray transform in n_e^2. Using Fourier transform in τ, inversion may be performed and one obtains explicitly:

$$n_e^2(\mathsf{D}, \zeta) = \int_{-\infty}^{\infty} \mathrm{d}\nu \frac{\mathrm{e}^{2\mathrm{i}\pi \frac{\nu}{\rho}\zeta}}{(J_l^L)^*(\nu/\rho)} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \mathrm{e}^{-2\mathrm{i}\pi\nu\tau} \frac{h_{\mathsf{S}}(\mathsf{D}, \tau)}{K(\tau)}, \tag{14}$$

and a similar expression for $n_e^2(\mathsf{S}, \zeta)$. Thus n_e can be exactly deduced from the r.h.s of Eq. (14). The accuracy of this solution depends on the validity of the “smallness” of ρ, which is fixed by the performance of the used measuring apparatus.

5. A class of special solutions by “decoupling”

Exploiting the structure of Eqs. (6), we construct a second class of special solutions in the so-called “decoupling” limit, whereby $n_e(\mathsf{S}, \zeta)$ is assumed known: either $n_e(\mathsf{S}, \zeta) = \text{constant}$ or $n_e(\mathsf{S}, \zeta) = y(\zeta)$ along a perpendicular line to the detector at site S. Then $n_e(\mathsf{D},\zeta)$ can be reconstructed elsewhere outside this line (i.e. at all sites D different from site S) as follows:

$$n_e(\mathsf{D},\zeta)=\int_{-\infty}^{\infty}\mathrm{d}u\,\rho^2\,\frac{Q_{\mathsf{S}}(\mathsf{D}|u\rho)}{H_l^L(u)}\,\mathrm{e}^{2\mathrm{i}\pi u\zeta},\quad \text{where } H_l^L(u)=\int_l^{L-\rho\tau}\frac{\mathrm{d}\zeta}{\zeta^2}y(\zeta)\,\mathrm{e}^{2\mathrm{i}\pi\zeta u}. \tag{15}$$

However this solution must be self-consistent and verify the second Eq. (6), which appears as a constraint on $y(\zeta)$:

$$h_{\mathsf{S}}(\mathsf{D}|\tau)=\frac{K(\tau)}{\rho^2}\int_l^{L-\rho\tau}\frac{\mathrm{d}\zeta_N}{\zeta_N^2}n_e(\mathsf{D},\zeta_N)y(\mathsf{S},\zeta_N+\rho\tau). \tag{16}$$

In the nonlinear integral equation approach self consistency of such solution is already taken into account. Thus fixing the electron density along some *line* allows to recover the electron density elsewhere in space [6] .

6. Concluding remarks

A new transformation called generalized X-ray transform is shown to arise from the analysis of the images given by double Compton-scattered radiation in transmission imaging. This transformation is *quadratic* and *bi-local* in the unknown function, the electronic density n_e. In this new transform information is collected on a pair of parallel lines instead of on a single line as in standard X-ray transform. This permits to generate a complete set of data (direct and permuted images) from which we derive equivalent nonlinear integral equations satisfied by *local* unknown functions. Physically, this transformation describes an exploration procedure whereby an incoming pencil beam probes the sample line by line and the data recorded on an energy sensitive detector.

Special classes of solutions are constructed by "linearization" and "decoupling". This theoretical result opens the way for a new defect detection method via n_e. In fact, the characterization of defects by n_e has several advantages: n_e is sensitive to composition changes in materials and allows its monitoring. It undergoes drastic variations through cracks, voids and holes and thus suited for defects detection.

This new imaging process, which uses doubly scattered radiation and yields the medium electronic density, may be regarded as complementary to the conventional imaging process, which operates with primary radiation and gives the linear attenuation coefficient of the medium μ. It is interesting to note that both functioning modalities can use the same transmission imaging apparatus. The use of double Compton scattered radiation permits the generation of images for the reconstruction of n_e without relative motion of the source-detector system. This is particular valuable in difficult operational situations.

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