It Takes Half the Energy of a Photon to Send One Bit Reliably on the Poisson Channel with Feedback

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Abstract—**We consider the transmission of a single bit over the continuous-time Poisson channel with noiseless feedback. We show that to send the bit reliably requires, on the average, half the energy of a photon. In the absence of peak-power constraints this holds irrespective of the intensity of the dark current. We also solve for the energy required to send $\log_2 \mathsf{M}$ bits.**

I. INTRODUCTION

The continuous-time Poisson channel models optical communication using direct detection. The input to the channel $x(\cdot)$ is nonnegative

$$x(t) \geq 0, \quad t \in \mathbb{R}, \tag{1}$$

and conditional on the input, the output $Y(\cdot)$ is a conditional Poisson process (also known as a doubly-stochastic Poisson process) of intensity $x(t) + \lambda_0$, where λ_0 is a nonnegative constant called *dark current*. Thus, conditional on the input, the output $Y(\cdot)$ is a nonhomogeneous Poisson process and thus of independent increments with

$$\Pr\big[Y(t+\tau) - Y(t) = \nu \,\big|\, \mathbf{X} = \mathbf{x}\big] = e^{-\Lambda} \frac{\Lambda^\nu}{\nu!}, \quad \nu \in \mathbb{Z}, \tag{2}$$

where

$$\Lambda = \int_t^{t+\tau} \big(x(\sigma) + \lambda_0\big) \, d\sigma. \tag{3}$$

To send a bit D taking on the values 0 and 1 equiprobably over this channel *without feedback* we use two input waveforms $x_0(\cdot)$ and $x_1(\cdot)$, and we send $x_0(\cdot)$ if $D = 0$ and $x_1(\cdot)$ if $D = 1$.

We refer to

$$\mathsf{E}\left[\int_t^{t+\tau} X(\sigma) \, d\sigma\right]$$

as the transmitted energy in the time interval $[t, t+\tau]$, although this is somewhat imprecise: this quantity is the expected number of transmitted photons in the interval, and one should technically multiply it by the energy in each photon (which depends on the light frequency) to obtain the transmitted energy in the interval.

We sometimes impose a *peak-power constraint* on the input, in which case we require that, with probability one,

$$x(t) \leq \mathsf{A}. \tag{4}$$

We then refer to A as the maximal allowed power (although, technically speaking, this needs to be normalized by the energy of each photon to have the sense of power.)

In the presence of feedback, the channel description is a bit more technical [1]. We require that conditional on $D = 0$, the channel output $Y(t)$ admit the $\mathcal{F}_t$ intensity $X_0(t) + \lambda_0$ [2, Chapter II, Section 3, Definition D7]. That is, conditional on $D = 0$, $Y(t)$ is a point process adapted to some history $\mathcal{F}_t$; $X_0(t)$ is a nonnegative $\mathcal{F}_t$-progressive process such that for all $t \geq 0$

$$\int_0^t X_0(s) \, ds < \infty;$$

and for all nonnegative $\mathcal{F}_t$-predictable processes $C(t)$

$$\mathsf{E}\left[\int_0^\infty C(t) \, dY(t)\right] = \mathsf{E}\left[\int_0^\infty C(t)\big(X_0(t) + \lambda_0\big) \, dt\right]. \tag{5}$$

The conditional expected energy transmitted when $D = 0$ over the time interval $[0, \mathsf{T}]$ is

$$\mathcal{E}_0 = \mathsf{E}\left[\int_0^\mathsf{T} X_0(t) \, dt\right]. \tag{6}$$

Similarly, when $D = 1$ the transmitted energy is

$$\mathcal{E}_1 = \mathsf{E}\left[\int_0^\mathsf{T} X_1(t) \, dt\right]. \tag{7}$$

The average transmitted energy is thus

$$\frac{1}{2}\big(\mathcal{E}_0 + \mathcal{E}_1\big). \tag{8}$$

A decoder is a mapping from the σ-algebra generated by $\{Y(t), \, 0 \leq t \leq \mathsf{T}\}$ to the set $\{0, 1\}$.

We say that a bit can be transmitted reliably over our channel with average transmitted energy $\mathcal{E}$, if for any $\epsilon > 0$ we can find some transmission interval T and a coding/decoding rule of expected transmission energy $\mathcal{E}$ and probability of error smaller than ϵ. We denote by $\mathcal{E}_{\min}$ the least energy required to transmit a bit reliably over our channel.

II. MAIN RESULT

Theorem 1. *The minimum energy required to send a single bit over the Poisson channel with dark current λ_0, feedback, and no peak-power constraint is*

$$\mathcal{E}_{\min} = \frac{1}{2}, \tag{9}$$

irrespective of the dark current. If the dark current is zero, then this is achievable even under a peak-power constraint whenever $\mathsf{A} > 0$.

III. Coding Scheme

The achievability when $\lambda_0 = 0$ is straightforward. To send $D = 0$ we transmit the all-zero input; to send $D = 1$ we transmit A until, through the feedback link, we learn that a count was registered; thereafter we send zero. The decoder guesses "$D = 0$" if no counts were registered in the interval $[0, \mathsf{T}]$, and guesses "$D = 1$" otherwise. Since $\lambda_0 = 0$, the probability of error given $D = 0$ is zero. Also, the transmitted energy when $D = 0$ is zero, so $\mathcal{E}_0 = 0$. Conditional on $D = 1$ the time of the first count is exponential with mean $1/\mathsf{A}$. Consequently, $\mathcal{E}_1 = 1$. Conditional on $D = 1$, the probability of error is the probability that no counts are registered in the interval $[0, \mathsf{T}]$. The probability of this event is the probability that the first count occurs after time T, i.e., the probability that a mean-$1/\mathsf{A}$ exponential exceeds T. It thus tends to zero as $\mathsf{T} \to \infty$.

If $\lambda_0 > 0$ and there is no peak-power constraint, we choose $\mathsf{A} \gg 1$ and $\Delta \ll 1$ and use the above scheme with $\mathsf{T} = \Delta$. We make sure that Δ is small enough for the probability of a spurious count in the interval $[0, \Delta]$ to be very small (the probability of a spurious count in this interval is $1 - e^{-\Delta\lambda_0}$), and we choose A large enough so that the probability that a mean-$1/\mathsf{A}$ exponential exceeds Δ is also very small.

IV. Converse

To prove that $\mathcal{E}_{\min}$ cannot be smaller than $1/2$, it suffices to consider the case where $\lambda_0 = 0$. We thus assume $\lambda_0 = 0$. In this case there is no loss in optimality in assuming that to send $D = 0$ we transmit the all-zero input. Indeed, given any general scheme consider the guess the decoder produces when faced with no counts. Call that FALSE. Let TRUE be its complement. Consider now a scheme with the same encoding rule for TRUE, with the same decoding rule, but where we send the all-zero input to convey FALSE. The new scheme uses less (or same) energy; has the same $p(\text{error}|\text{TRUE})$; and has $p(\text{error}|\text{FALSE}) = 0$. Since the name we give to the hypotheses is immaterial, we can assume that FALSE corresponds to $D = 0$.

Next we argue that there is no loss in optimality in restricting ourselves to a detector that bases its decision on the presence of counts in the interval $[0, \mathsf{T}]$. Since this enlarges the set of outcomes yielding the guess "D=1", this cannot increase $p(\text{error}|D = 1)$. To send $D = 0$ we send the all-zero waveform, which results in no counts (there is no dark current), so this does not change $p(\text{error}|D = 0)$.

Finally, we argue that there is no loss in optimality in stopping transmission once a count has been registered. Indeed, this reduces the transmitted energy and does not change the performance of the above detector.

We next analyze the probability of error of such schemes. Conditional on $D = 0$, the probability of error is zero, because there is no dark current so sending zero input guarantees zero

counts. As to the conditional probability of error given $D = 1$, let T_1 denote the random time at which the first count is registered. Substituting the stochastic process

$$C(\omega, t) = \mathrm{I}\{t \leq T_1(\omega) \wedge \mathsf{T}\}, \quad (\omega, t) \in \Omega \times [0, \infty). \quad (10)$$

in (5) yields

$$
\begin{aligned}
p(\text{correct}|D = 1) &= \Pr[T_1 \leq \mathsf{T} \,|\, D = 1] \\
&= \mathsf{E}\big[Y(T_1 \wedge \mathsf{T}) \,\big|\, D = 1\big] \\
&= \mathsf{E}\left[\int_0^\infty C(t)\,\mathrm{d}Y(t) \,\bigg|\, D = 1\right] \\
&= \mathsf{E}\left[\int_0^\infty C(t)X_1(t)\,\mathrm{d}t\right] \\
&= \mathsf{E}\left[\int_0^{T_1 \wedge \mathsf{T}} X_1(t)\,\mathrm{d}t\right] \\
&= \mathcal{E}_1.
\end{aligned}
$$

For the probability of error to tend to zero, the expected energy transmitted to convey $D = 1$ must thus approach 1.

V. Sending $\log_2 \mathsf{M}$ Bits

More generally, to send $\log_2 \mathsf{M}$ bits requires

$$\frac{\mathsf{M} - 1}{\mathsf{M}} \quad (11)$$

of the energy of a photon. When $\mathsf{M} = 2$ we recover the required energy to send one bit. To prove the converse—that one cannot accomplish the task with less energy—requires a simple genie-aided argument. Once again we can assume no dark current, and we can show that there is no loss in optimality in conveying the zero message using the all-zero input and by limiting ourselves to detectors that guess that the transmitted message was the zero message if, and only if, no counts were registered. We then consider a genie that, if a count is registered, tells the detector which message was sent. With the aid of the genie the detectors errs if, and only if, the transmitted message was not the zero message and no counts were registered. Thus, by our previous analysis, the required energy of each of the nonzero messages must be one. Averaging over the equally-likely messages demonstrates that $(\mathsf{M} - 1)/\mathsf{M}$ of the energy of a photon is necessary for reliable transmission.

The direct part in the absence of dark current is based on a simple scheme where the transmission interval $[0, \mathsf{T})$ is divided into $\mathsf{M} - 1$ intervals. We associate with the nonzero message $m \in \{1, \ldots, \mathsf{M} - 1\}$ the interval

$$\big[(m - 1)\mathsf{T}/(\mathsf{M} - 1), m\mathsf{T}/(\mathsf{M} - 1)\big).$$

To send the nonzero message m we transmit with a very high power starting at time $(m - 1)\mathsf{T}/(\mathsf{M} - 1)$ until a count is registered or until the end of the interval at time $m\mathsf{T}/(\mathsf{M} - 1)$. To send the zero message we send the all-zero signal. The detector operates as follows. If no counts are registered, it guesses that the zeroth message was sent. If a count is registered, it declares that the transmitted message was the

one that corresponds to the interval in which the count was registered.

In the presence of dark current we use the same scheme except that we choose as our transmission time a very short interval $[0, \Delta]$ in which the probability of a spurious count is negligible.

VI. DISCUSSION

In the absence of a peak power constraint, the capacity of the Poisson channel (with or without feedback) is infinite [3], [4]. Thus, in sending a very large number of bits, reliable communication can be had with an arbitrarily small expected energy per bit. The situation changes dramatically when sending a *single* bit. Even in the presence of feedback, the required energy is finite; it is, in fact, $1/2$.

This should be contrasted with the infinite-bandwidth Gaussian channel, where a single bit can be sent reliably with the same amount of energy that would be required per bit if one were sending a large number of bits [5], [6].

However, if the allowed energy is that of one photon, then we can send as many bits as we want with arbitrarily small probability of error.

REFERENCES

[1] A. Lapidoth, "On the reliability function of the ideal Poisson channel with noiseless feedback," *IEEE Trans. on Inform. Theory,* vol. IT-39, pp. 491–503, March 1993.

[2] P. Brémaud, *Point Processes and Queues: Martingale Dynamics,* Springer-Verlag, New York, 1981.

[3] Y. Kabanov, "The capacity of a channel of the Poisson type," *Theory of Probability and Its Appl.*, vol. 23, pp. 143–147, 1978.

[4] M. H. A. Davis, "Capacity and cutoff rate for Poisson-type channels," *IEEE Trans. Inf. Theory*, vol. 26, no. 6, pp. 710–715, Nov. 1980.

[5] G. L. Turin, "Signal design for sequential detection systems with feedback," IT-11, pp. 401–408, July 1965.

[6] Y. Polyanskiy, H.V. Poor, and S. Verdú, "Minimum energy to send k bits with and without feedback," ISIT 2010.