Multi-Rate Resource Allocations for TH-UWB Wireless Communications

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Abstract—In this paper, we are interested in resource allocation strategies for wireless time-hopping ultra-wide band (TH-UWB) communications with multiple rate capabilities between users. Multiple rates are achieved by assigning different processing gains, i.e. N_f, to users. For this purpose, the multiple-access interference (MAI) variance accounting for multi-rate is needed. It is a challenging task due to the lack of a suitable closed-form expression for the MAI variance in a multi-rate context. We further study the multi-rate resource allocation problem in uplink TH-UWB systems for which an optimal search cannot be envisaged due to the exponential complexity induced. Our contribution lies in three-fold: i) A new intercode correlation expression accounting for multi-rate communications is derived, and the variance of the MAI averaging over the codes is obtained. ii) The multi-rate resource allocation problem is tackled by relaxing the integer constraint on the processing gains and modeled via a signomial programming problem. iii) Based on this, a branch and bound (BB) algorithm is derived for the allocation of the processing gains in TH-UWB systems. We also propose a really simple heuristic with linear complexity for the N_f allocation. We show that the algorithm proposed outperforms the BB algorithm in average throughput and average starvation rate [1].

I. INTRODUCTION

Time-hopping ultra-wideband (TH-UWB) based on impulsive radio technology is a very promising technique to achieve high spectral efficiency under low radiated power [2]. This technique has received a great amount of attention from the scientific community during the last decade [3], [4]. TH-UWB is a code division multiple access (CDMA) technology; multiple users can access to the channel at the same time by assigning a time-hopping code (THC) to each user. In [5], the authors showed that an optimal THC design rule can be derived by minimizing the variance of the multiple access interference (MAI), but without considering the general multi-rate case.

Resource allocation is a critical task in the cellular system optimization, greatly influencing the network performance. In this paper, we are interested in resource allocation strategies allowing to increase the global data rate in the uplink scenario. The power control cannot really be envisaged in practical TH-UWB systems, due to the very large bandwidth of the system,

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e.g. several Gigahertz. Moreover, it has been shown that rate control is an efficient technique to achieve high throughput in TH-UWB systems. In [6], the authors have shown that the coding rate adaptation allows to achieve better throughput than power adaptation if the rate control is performed according to the interference level experienced at the destination. However, they do not consider the rate adaptation based on a variable symbol length which is the case when the THC have different processing gains.

Since TH-UWB systems are based on CDMA technology, works on rate allocation in multi-rate cellular CDMA systems are pertinent for our study, e.g. [7], [8] and references therein. Authors in [7] consider the adaptive rate allocation problem in DS-CDMA systems, by assigning various spreading gains among users. However, the particular frame structure of TH-UWB systems with N_c chips and N_f frames implies non trivial dependency between these parameters and the global throughput. This particular structure makes the physical (PHY) layer model of [7] as well as the associated multi-rate resource allocation strategy unsuitable for TH-UWB systems. On the other hand, the works dealing with multi-rate TH-UWB systems do not focus on rate adaptation via a variable processing gain allocation. Indeed, the authors in [9] developed an SINR model for multi-rate TH-UWB systems based on an approximation of the MAI variance. However, only AWGN and synchronous transmission have been considered and hence the variance expressions given in [9] are simpler and not as realistic as the ones which would be obtained in multipath fading environments. Moreover, they considered that the processing gain ratio between users is an integer. This is a strong hypothesis, significantly simplifying the intercode interference analysis. In [10] the authors deal with a maximum-likelihood (ML) receiver for multi-rate TH-UWB communications and the work in [11] deals with coded and uncoded TH-UWB systems with multi-rate capabilities with multi-services assignment. The authors in [11] effectively consider the use of various spreading gains for several rate services. However, they only consider the AWGN channel case and no spreading gain allocation has been studied.

In this work, we consider the general case of multipath fading channels and non-integer processing gain ratio between users. An accurate MAI model based on [5] is derived for multi-rate TH-UWB communication systems and the proofs of theorems given in [1] are provided. These proofs significantly enhance our previous paper since the intercode correlation in the general multi-rate context was unknown and far from trivial. We extend the problem of variable spreading gain allocation as partially treated in [1] by showing that the

processing gain allocation is a general mixed integer and signomial programming problem. Due to its very difficult nature, it cannot be guaranteed to find the global optimal solution. We prove that the mixed integer and signomial programming problem can be approached locally by a posynomial problem and hence can be solved via the combination of geometric programming and a branch and bound (BB) algorithm. These new algorithms can serve as benchmarks for the evaluation of other algorithms and/or heuristics. Finally, we compare the performance of a new heuristic with linear complexity, partially presented in [1], w.r.t. the BB-based algorithm.

The remainder of the paper is organized as follows. The next section introduces the system model. Section III provides a new closed form expression for the variance of the multiple access interference in multi-rate TH-UWB communications. In Section IV, we revisit the allocation of the spreading gains as a signomial programming problem. We hence provide a solution by relaxing the integer constraint on the N_f values and by approaching locally the signomial problem by a posynomial problem. We further deal with the integer constraint by proposing a BB algorithm based on the previous formulation and we present a simpler heuristic with linear complexity to allocate the spreading gains to the users. Section V gives the numerical results by comparing the BB performances to the proposed heuristic and Section VI draws the conclusions.

II. SYSTEM MODEL

We consider asynchronous uplink multiuser communications in a single cell network, with one base station (BS) and N_u users. A UWB symbol is defined as N_f frames each containing N_c chips. The number of chips per frame, i.e. N_c, and the duration of the chip, T_c, are fixed for all users in the network. The UWB symbol duration of the $u-$th user is $T_s^{(u)} = N_c N_f^{(u)} T_c$, with $N_f^{(u)}$ the number of frames of the $u-$th user. The signal transmitted by the $u-$th user is:

$$s_u(t) = \sum_i d_u(i) \sum_{j=0}^{N_c N_f^{(u)} - 1} c_u(j) w \left(t - i T_s^{(u)} - j T_c - \theta_u \right),$$
(1)

where $w(t)$ is the impulse of duration $T_w \ll T_c$, d_u are the transmitted PAM information symbols with $\mathbb{E}\left[d_u^2\right] = 1$ and θ_u is the asynchronism between users. Moreover, $c_u := \{c_u(j)\}_{j=0}^{N_c N_f^{(u)} - 1}$ is the $u-$th developed time hopping code (DTHC) as defined in [5]. The UWB signal is sent through a multipath channel with N_p paths and processed at the BS by a rake receiver containing L_r fingers. The intersymbol interference (ISI) can be neglected by inserting a guard time at the end of each frame [3], [5], [12]. If the user u is assumed to be of interest, its received signal at the BS can be decomposed as [5]:

$$z^{(u)} = z_u + z_{mai} + \eta^{(u)},$$
(2)

with:

$$z_u = \sqrt{P_u} \sum_{l=1}^{L_r} \left(A_u^l \right)^2 N_f^{(u)} d_u(0),$$
(3)

$$z_{mai} = \sum_{l=1}^{L_r} A_u^l \sum_{\substack{u'=1 \\ u' \neq u}}^{N_u} \sqrt{P_{u'}} \sum_{n=1}^{N_p} A_{u'}^n y_{u',u}^{n,l}(\theta_{u'}),$$
(4)

where:

$$y_{u',u}^{n,l}(\theta_{u'}) = \sum_i d_{u'}(i) \sum_{j=0}^{N_c N_f^{(u')} - 1} \sum_{j_u=0}^{N_c N_f^{(u)} - 1} c_{u'}(j) c_u(j_u)$$
$$\times r_{ww} \left(i T_s^{(u')} + (j - j_u) T_c + \Delta_{u',u}^{n,l} + \theta_{u'} \right),$$
(5)

z_u and z_{mai} are the useful part of the signal and the multiple access interference respectively. Moreover, $r_{ww}(s) = \int_{-\infty}^{\infty} w(t) w(t - s) dt$ and $\eta^{(u)}$ is the filtered Gaussian noise with N_0 as the one-sided power spectral density and its expression can be found in [5], [12]. $A_u^n = a_u^n e^{-\tau_u^n / 2\gamma}$ is the $n-$th path amplitude of the $u-$th user where τ_u^n is the delay of the $n-$th path of the user u and a_u^n are zero mean random variables (RVs) independent of delays and with a variance σ_a^2 [5], [13]. Moreover γ is a statistical channel parameter and is related to the channel impulse response length as defined in [13] and used in [5], [12]. We also define $\Delta_{u',u}^{n,l} = \tau_{u'}^n - \tau_u^l$ and P_u is the received power at the BS for the $u-$th user after path loss propagation.

We set $N_f^{(u)} = \alpha^{(u')} N_f^{(u')}$, with $\alpha^{(u')} > 0$ and $\alpha^{(u')} \in \mathbb{Q}$, i.e. the rational number set. As proposed in [5], we consider the Euclidean division of $\theta_{u'} + \Delta_{u',u}^{n,l}$ w.r.t. $T_s^{(u')}$ and T_c yielding to $\theta_{u'} + \Delta_{u',u}^{n,l} := Q_{u'}^{n,l} T_s^{(u')} + q_{u'}^{n,l} T_c + \epsilon_{u'}^{n,l}$ with:

$$Q_{u'}^{n,l} = \left\lfloor \frac{\theta_{u'} + \Delta_{u',u}^{n,l}}{T_s^{(u')}} \right\rfloor \in \{-\infty, \infty\},$$
(6)

$$q_{u'}^{n,l} = \left\lfloor \frac{\theta_{u'} + \Delta_{u',u}^{n,l} - Q_{u'}^{n,l} T_s^{(u')}}{T_c} \right\rfloor \in \left\{ 0, \cdots, N_c N_f^{(u')} - 1 \right\}$$
(7)

and $\epsilon_{u'}^{n,l} \in [0, T_c[$ is the remainder of the Euclidean division and $\lfloor \cdot \rfloor$ denotes the floor rounding. Thanks to this relationship, eq. (5) can be written as:

$$y_{u',u}^{n,l}(\theta_{u'}) = \sum_i d_{u'}(i) \sum_{j=0}^{N_c N_f^{(u')} - 1} \sum_{j_u=0}^{N_c N_f^{(u)} - 1} c_{u'}(j) c_u(j_u)$$
$$\times r_{ww} \left(\left(i + Q_{u'}^{n,l} \right) T_s^{(u')} + \left(q_{u'}^{n,l} + j - j_u \right) T_c + \epsilon_{u'}^{n,l} \right)$$
(8)

III. VARIANCE OF z_{mai} WITH MULTIPLE RATES

A. Expression of multiple access interference

We can prove that the autocorrelation function r_{ww} in (8) is non zero if and only if $-2 < Q_{u'}^{n,l} + i \leq \left\lceil \alpha^{(u')} \right\rceil - 1$, with $\lceil \cdot \rceil$ being the ceil rounding. Our first theoretical result is stated in the following lemma [1], which extends and generalizes the result in [5]:

Lemma 1 *In multi-rate PAM TH-UWB communications, the multiuser interference can be written as:*

$$y_{u',u}^{n,l}(\theta_{u'}) = \sum_{k=-1}^{\lceil \alpha^{(u')} \rceil - 1} d_{u'}\left(k - Q_{u'}^{n,l}\right) \times$$
$$\left[C_{u,u'}^{k}\left(q_{u'}^{n,l}\right) r_{ww}\left(\epsilon_{u'}^{n,l}\right) + \right.$$
$$\left. C_{u,u'}^{k}\left(q_{u'}^{n,l} + 1\right) r_{ww}\left(\epsilon_{u'}^{n,l} - T_c\right)\right] \quad (9)$$

Where $\forall \alpha^{(u')}$:

$$C_{u,u'}^{-1}(q) = \sum_{p=0}^{\min\left(q, N_c N_f^{(u)}\right) - 1} c_u(p)\, c_{u'}(p - q) \quad (10)$$

If $\alpha^{(u')} < 1$:

$$C_{u,u'}^{0}(q) = \sum_{p=\min\left(q, N_c N_f^{(u)}\right)}^{N_c N_f^{(u)} - 1} c_u(p)\, c_{u'}(p - q) \quad (11)$$

If $\alpha^{(u')} \geq 1$ *and for* $0 \leq k \leq \lceil \alpha^{(u')} \rceil - 1$ *then:*

$$C_{u,u'}^{k}(q) = \sum_{p=q+kN_c N_f^{(u')}}^{\min\left(q+(k+1)N_c N_f^{(u')}, N_c N_f^{(u)}\right) - 1} c_u(p) \times$$
$$c_{u'}\left(p - q - kN_c N_f^{(u')}\right) \quad (12)$$

The proof is provided in Appendix VII-A.

B. Variance of z_{mai} with multiple rate

Thanks to the Lemma 1 and by averaging over the amplitudes A, symbols d_u, asynchronism θ_u and delays τ_u as in [5], the variance of the MAI w.r.t. the THC can be written as:

$$V_{mai|c}^{(u)} = \Lambda \sum_{\substack{u'=1 \\ u' \neq u}}^{N_u} \frac{P_{u'}}{N_f^{(u')}} \sum_{q=0}^{N_c N_f^{(u')} - 1} \sum_{k=-1}^{\lceil \alpha^{(u')} \rceil - 1} \left(C_{u,u'}^{k}(q)\right)^2,$$
$$(13)$$

with $\Lambda = \rho_{ww}(0)\sigma_a^4 \sum_{l=1}^{L_r} \sum_{n=1}^{N_p} (\lambda/(\lambda + 1/\gamma))^{n+l} / \left(3N_c^4 T_c\right)$ where $\rho_{ww}(0) = \int_{-\infty}^{\infty} r_{ww}^2(t)dt$ and λ is the channel path density [5], [13]. In order to express the global average multiple access interference, we need to average over the codes and the following theorem holds [1]:

Theorem 1 (Variance of z_{mai}) *In multi-rate TH-UWB communications with PAM signals, the variance of the MAI averaging over the codes is:*

$$V_{mai}^{(u)} = \Lambda \sum_{\substack{u'=1 \\ u' \neq u}}^{N_u} \frac{P_{u'}}{N_f^{(u')}} M_c(u, u'), \quad (14)$$

with, if $\alpha^{(u')} \geq 1$ *then* $M_c(u, u') := M_c^{+}(u, u')$:

$$M_c^{+}(u, u') = N_f^{(u')}\left[3N_c\left(N_c^2 + N_c N_f^{(u')} - 1\right) N_f^{(u)} \right.$$
$$\left. - N_c^2 N_f^{(u')^2} + 1\right], (15)$$

and if $\alpha^{(u')} < 1$ *then* $M_c(u, u') := M_c^{-}(u, u')$:

$$M_c^{-}(u, u') = N_f^{(u)}\left[3N_c\left(N_c^2 + N_c N_f^{(u)} - 1\right) N_f^{(u')} \right.$$
$$\left. - N_c^2 N_f^{(u)^2} + 1\right]. (16)$$

A sketch of proof is provided in Appendix VII-B.

From Lemma 1 and Theorem 1, the signal to interference and noise ratio (SINR) of the user u can be written as in eq. (17) at the top of the next page. We define $G_u = \mathbb{E}_{a,\tau}\left[\sum_{l=1}^{L_r} \left(A_u^l\right)^2\right]$ and $V_n = \sigma_a^2 \sum_{l=1}^{L_r} (\lambda/(\lambda + 1/\gamma))^l$ is the noise enhancement due to the rake receiver, moreover N_0 is the one-sided noise power spectral density (PSD). We also define the following sets [1]: $I_+ = \left\{u' \mid \alpha^{(u')} \geq 1, u' \neq u\right\}$ and $I_- = \left\{v \mid \alpha^{(v)} < 1\right\}$, such as $I_+ \cup I_- = I$ and $I_+ \cap I_- = \emptyset$. Fig. 1 illustrates the second order moments of the intercode correlation w.r.t. the delay q of the intercorrelation for $\alpha^{(u')} = 3/8 < 1$ (Fig. 1(a)) and for $\alpha^{(u')} = 8/3 \geq 1$ (Fig. 1(b)). When $\alpha^{(u')} = 3/8$, according to Lemma 1 there are two intercorrelation terms, i.e. $C_{u,u'}^{-1}(q)$, $C_{u,u'}^{0}(q)$ and four terms, i.e. $C_{u,u'}^{-1}(q)$, $C_{u,u'}^{0}(q)$, $C_{u,u'}^{1}(q)$, $C_{u,u'}^{2}(q)$ for $\alpha^{(u')} = 8/3$. The THC are randomly selected from a binomial random variable, it means the pulse position in the code structure is selected randomly, according to a Bernoulli variable for each pulse (cf. the proof of the Theorem 1 and [14]). Fig. 2 shows the SINR of the user 1, assumed to be the user of interest, evaluated with eq. (17) compared to the SINR in simulation w.r.t. the number of users. The number of frames of the user 1 is $N_f^{(1)} = 8$ and the number of frames of the interfering users are respectively: $N_f^{(2)} = 3$ for the user 2, $N_f^{(3)} = 4$ for the user 3 and so on until $N_f^{(11)} = 12$ for the user 11. A perfect agreement between the theory and the simulation can be observed in Figs. 1 and 2 which validates our findings.

In Fig. 3, the average SINR is plotted for user 1 assumed to be the user of interest and considering another interfering user in the network, i.e. user 2. The SINR is plotted according to some values of the number of frames of the interfering user, i.e. $N_f^{(2)} = 1, 3, 5, 13$. The number of chips is fixed to $N_c = 13$ and the chip duration is $T_c = 5$ ns. The channel model used is the one described in [5] with $\lambda = 2.1$ ns^{-1}, $\gamma = 12$ ns and $N_p = 25$. One can observe that the SINR of user 1 increases as $N_f^{(1)}$ increases as expected because of the useful power dependence on $N_f^{(1)^2}$. We also observe a higher SINR sensibility to the number of frames of the interferer for higher values of $N_f^{(1)}$ than for lower values. Since we have provided theoretical background for multiuser multi-rate SINR, we will now move on the suitable processing gain allocation.

IV. ADAPTIVE RATE ALLOCATION SCHEMES

In this section, we study the multiple rate allocation problem in order to maximize the global throughput for TH-UWB systems in the uplink scenario. First, the integer constraint on $N_f^{(u)}$ is relaxed yielding to a signomial programming problem. In a second step, the integer constraint is taken into account and the optimization problem is solved via a BB algorithm. We finally propose a simpler heuristic for the adaptive rate allocation.

$$\text{SINR}_u\left(N_f^{(u)}\right) = \frac{P_u G_u N_f^{(u)2}}{\Lambda\left(\sum_{\substack{u' \in I_+ \\ u' \neq u}} \frac{P_{u'}}{N_f^{(u')}} M_c^+\left(u,u'\right) + \sum_{v \in I_-} \frac{P_v}{N_f^{(v)}} M_c^-\left(u,v\right)\right) + \frac{N_0 N_f^{(u)}}{2} V_n} \tag{17}$$
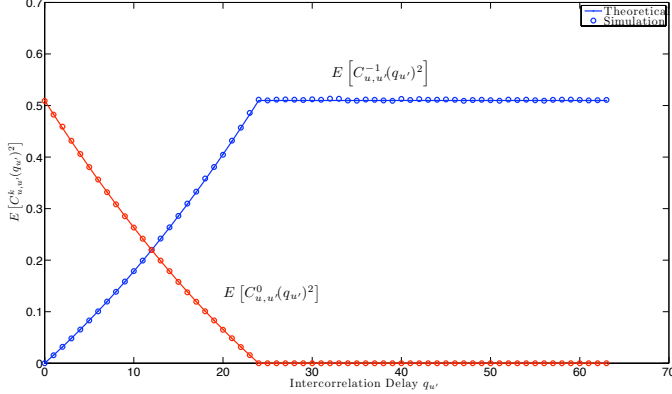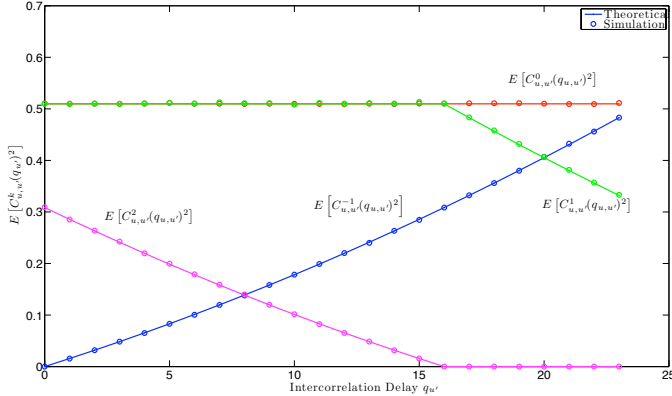
Fig. 1. Second order moments of intercode cross-correlation functions for $\alpha^{(u')} < 1$ and for $\alpha^{(u')} \geq 1$
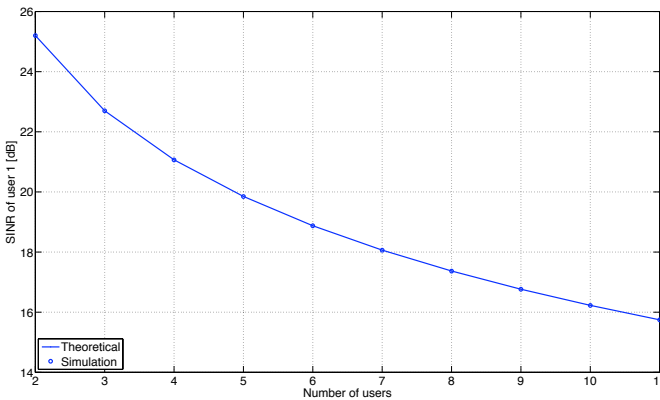
Fig. 2. Average SINR for user 1 w.r.t. the number of users. The number of frames of the user 1 is $N_f^{(1)} = 8$
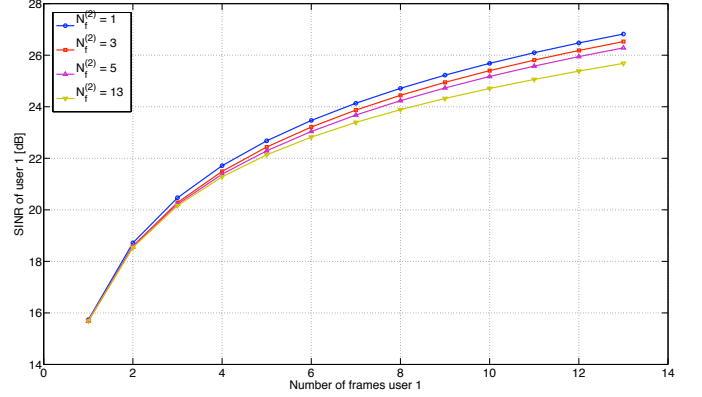
Fig. 3. Average SINR for user 1 considering two active users versus the number of frames of the user 1 and labeled on $N_f^{(2)}$

A. Signomial Programming

The effective throughput of a user u is $D_u = 1/\left(N_c N_f^{(u)} T_c\right)$ provided that its SINR is greater than a threshold $\Gamma_{\min}$ [1], [7]. The maximization of the global throughput, i.e. $\max \sum_u D_u$, subject to an SINR constraint for each user can be written as in [1]. This problem is highly non-convex, essentially because of the SINR requirements, and combines continuous constraints (i.e. SINR constraints) and integer constraints (i.e. $N_f^{(u)} \in \{1, \cdots, N_c\}$). This combination induces an exponential complexity, i.e. $O\left((N_c)^{N_u}\right)$, of the optimal search and cannot be envisaged for large problem (i.e. large N_c and N_u).

Actually, the cost function of the optimization problem in [1], i.e. $\max \sum_{u=1}^{N_u} 1/N_f^{(u)}$ can be easily proved to be equivalent to $\min \prod_{u=1}^{N_u} N_f^{(u)}$ yielding to the modified optimization problem:

$$
\begin{aligned}
&\min_{\mathbf{N_f}} \quad \prod_{u=1}^{N_u} N_f^{(u)}, \text{ s.t.} \\
&(c_1) \quad \text{if } P_u > 0 \text{ then } \frac{\Gamma_{\min}}{\text{SINR}_u\left(N_f^{(u)}\right)} \leq 1, \forall u \in I \\
&(c_2) \quad N_f^{(u)} \geq 1, \forall u \in I, \\
&(c_3) \quad N_f^{(u)} \leq N_c, \forall u \in I, \\
&(c_4) \quad N_f^{(u)} \in \mathbb{N}, \forall u \in I,
\end{aligned} \tag{18}
$$

where I is the set of the transmitting users, $\mathbf{N_f} = \left[N_f^{(1)}, \cdots, N_f^{(N_u)}\right]^T$ is a vector representing the number of frames for each user. The optimization problem stated in (18) is a mixed integer signomial programming (MISP) problem for which a global optimal solution cannot be found efficiently [15]. The problem is referred as signomial because of summation of products with positive and negative coefficients in the constraint (c_1) [15]. The constraint (c_3) refers to the work

$$\beta_u = \frac{r_u\left(\hat{\mathbf{N}}_\mathbf{f}\right)}{\prod_{j=1}^{N_u}\left(N_f^{(j)}\right)^{\alpha_u}} \quad \text{and} \quad \alpha_u = \frac{N_c^2\Gamma_{\min}(u)\Lambda\left[\sum_{v\in I_-}P_v N_f^{(v)^{-1}}N_f^{(u)} - 2\sum_{u'\in I_+}P_{u'}N_f^{(u')^2}N_f^{(u)^{-2}}\right]}{r_u\left(\hat{\mathbf{N}}_\mathbf{f}\right)} \quad \forall u \in I \ (19)$$

$$\begin{aligned} p_u\left(\hat{\mathbf{N}}_\mathbf{f}\right) &= 3N_c^2\Gamma_{\min}(u)\Lambda\sum_{u'\in I_+}P_{u'}N_f^{(u')}N_f^{(u)^{-1}} + 3\Gamma_{\min}(u)\Lambda N_c\left(N_c^2-1\right)\left(\sum_{u'\in I_+}P_{u'} + \sum_{v\in I_-}P_v\right)N_f^{(u)^{-1}} \\ &+ \Gamma_{\min}(u)\Lambda\sum_{u'\in I_+}P_{u'}N_f^{(u)^{-2}} + \Gamma_{\min}(u)\Lambda\sum_{v\in I_-}P_v N_f^{(v)^{-1}}N_f^{(u)^{-1}} + 3\Gamma_{\min}(u)\Lambda N_c^2\sum_{v\in I_-}P_v + \Gamma_{\min}\frac{N_0 V_n}{2}N_f^{(u)^{-1}} \end{aligned}$$
$$(20)$$

of Le Martret *et al.* in [5] who have shown that there exists rigorous algebraic conditions to minimize the multiple access variance of pair of codes and in particular N_f should not be greater than N_c. The problem in (18) can be converted locally in a geometric programming optimization problem by relaxing the integer constraint on $N_f^{(u)}$.

Proposition 1 *The MISP optimization problem stated in* (18) *can be locally approximated by the following geometric programming problem (GPP):*

$$\min_{\mathbf{N}_\mathbf{f}} \quad \prod_{u=1}^{N_u} N_f^{(u)}, \ s.t.$$

$(\tilde{c}_1)$ *if $P_u > 0$ then $\frac{p_u(\hat{\mathbf{N}}_\mathbf{f})}{\beta_u\prod_{j=1}^{N_u}\left(\hat{N}_f^{(j)}\right)^{\alpha_u}} \leq 1, \forall u \in I$*

(c_2) $N_f^{(u)} \geq 1, \forall u \in I,$

(c_3) $N_f^{(u)} \leq N_c, \forall u \in I,$

$$(21)$$

$\beta_u\prod_{j=1}^{N_u}\left(\hat{N}_f^{(j)}\right)^{\alpha_u}$ *is the best local monomial approximation around $\hat{\mathbf{N}}_\mathbf{f}$ of the posynomial $r_u\left(\hat{\mathbf{N}}_\mathbf{f}\right)$ given as:*

$$\begin{aligned} r_u\left(\hat{\mathbf{N}}_\mathbf{f}\right) &= 1 + N_c^2\Gamma_{\min}(u)\Lambda\sum_{u'\in I_+}P_{u'}N_f^{(u')^2}N_f^{(u)^{-2}} + \\ & \quad N_c^2\Gamma_{\min}(u)\Lambda\sum_{v\in I_-}P_v N_f^{(v)^{-1}}N_f^{(u)} \end{aligned}$$

where $\Gamma_{\min}(u) = \Gamma_{\min}/\left(P_u G_u\right)$ and the parameters β_u and α_u of the monomial approximation are given in eq. (19) *on the top of the page. Moreover, $p_u\left(\hat{\mathbf{N}}_\mathbf{f}\right)$ is a posynomial whose the expression is given in eq.* (20) *on the top of the page.*

A proof is given in Appendix VII-C.

The constraint $(\tilde{c}_1)$ in Proposition 1 is now a posynomial constraint (the ratio between a posynomial and a monomial is a posynomial). Hence, if $N_f^{(u)}$ is allowed to be real valued in $[1, N_c]$, the optimization problem in (21) is a geometric programming problem and can be solved very efficiently with modern techniques [15]. This allows us to find a local optimal allocation of $N_f^{(u)}$ $\forall u \in I$. However, since (21) is only a local approximation of the problem (18), the optimal point

$\hat{\mathbf{N}}_\mathbf{f}$ returns by the resolution of (21) cannot be considered as valid if it is too far from the current guess [15]. Hence, additive constraints on the validity of the solution need to be added, leading to an iterative resolution of (21). It can be stated by the following problem:

$$\min_{\mathbf{N}_\mathbf{f}} \quad \prod_{u=1}^{N_u} N_f^{(u)}, \ s.t.$$

$(\tilde{c}_1)$ if $P_u > 0$ then $\frac{p_u(\hat{\mathbf{N}}_\mathbf{f})}{\beta_u\prod_{j=1}^{N_u}\left(\hat{N}_f^{(j)}\right)^{\alpha_u}} \leq 1, \forall u \in I$

(c_2) $N_f^{(u)} \geq 1, \forall u \in I,$

(c_3) $N_f^{(u)} \leq N_c, \forall u \in I,$

$(\tilde{c}_4)$ $(1-\eta)\hat{N}_f^{(u)} \leq N_f^{(u)} \leq (1+\eta)\hat{N}_f^{(u)}, \forall u \in I$

$$(22)$$

where η controls the validity of the next guess; it ensures the next estimation of the solution to be near to the current guess, i.e. $\hat{\mathbf{N}}_\mathbf{f}$. In order to solve the optimization problem stated in (22), we propose the adaptive rate allocation with signomial programming (ARASP) procedure stated in the algorithm 1. The function solveGP is a procedure solving geometric problems very efficiently with traditional convex solver tools [15], [16] [2]. The algorithm starts once a feasible processing gain vector is found, i.e. a vector $\mathbf{N}_\mathbf{f}$ making the problem (22) feasible. Once the problem is feasible, the algorithm iterates until the convergence (step 8). It is worth noting that even though the problem was feasible, it can become infeasible, i.e. step 11. Indeed, not only the problem in (18) is signomial but it also contains high non-linearities in the SINR constraints which can make the problem infeasible even though it was feasible at first. Let us focus on this issue for a while and in particular on the SINR expression in (17). We remark that two subsets I_+ and I_- are involved at the denominator. According to the range of these subsets the SINR expression changes. Moreover, these subsets are defined according to the $N_f^{(u')}$ values of the interfering users u' compared to the $N_f^{(u)}$ of the user of interest u. It means that while the $N_f^{(u)}$ value is updated for each user, the SINR expression

[2]We have used the tools developed by Boyd *et al* in order to solve the convex problem related to (22) available at http://www.stanford.edu/ boyd/index.html

is changing as well as the constraints ($\tilde{c}_1$) in (22). In other words, the coefficients of the posynomial constraints ($\tilde{c}_1$) in (22) are changing from one iteration to another implying non linearities in the algorithm. This property is very critical and prevents to find the global optimal solution surely. However, the proposed algorithm approaches the optimal solution by a local approximation of the signomial constraint combined with an iterative procedure in order to converge toward a suitable solution.

Algorithm 1 Adaptive Rate Allocation with Signomial Programming (ARASP)

Require: $N_u \geq 1$, $N_c > 1$,
Ensure: $\mathbf{N_f} \in \mathbb{R}$ allocation for feasible problems
 1: Initialize M, η, ϵ, $i = 0$ and find a feasible $\hat{\mathbf{N}}_{\mathbf{f}}$. $A = \{u \,|\, \mathrm{SINR}_u < \Gamma_{\min}\}$
 2: $\mathbf{N_f}(\mathbf{1}) \leftarrow \hat{\mathbf{N}}_{\mathbf{f}}$
 3: $[\hat{\mathbf{N}}_{\mathbf{f}}, \mathrm{status}] \leftarrow \mathrm{solveGP}\left(\mathbf{N_f}(\mathbf{1}), \mathrm{SINR}_u\right)$
 4: **if** status = infeasible **then**
 5:    return $\mathbf{N_f}(\mathbf{1})$ and quit
 6: **else if** status = solved **then**
 7:    update SINR_u and A with $\hat{\mathbf{N}}_{\mathbf{f}}$
 8:    **while** $\left(\max_u \left|\hat{\mathbf{N}}_{\mathbf{f}} - \mathbf{N_f}\right| > \epsilon \,|\, A \neq \emptyset\right) \& \ i \leq M$ **do**
 9:      $\mathbf{N_f} \leftarrow \hat{\mathbf{N}}_{\mathbf{f}}$
10:      $[\hat{\mathbf{N}}_{\mathbf{f}}, \mathrm{status}] \leftarrow \mathrm{solveGP}\left(\mathbf{N_f}, \mathrm{SINR}_u\right)$, $i = i + 1$
11:      **if** status = infeasible **then**
12:        $\hat{\mathbf{N}}_{\mathbf{f}} \leftarrow \mathbf{N_f}$, $i = M + 1$
13:      **end if**
14:      update SINR_u and A
15:    **end while**
16:    **if** status = infeasible **then**
17:      return $\mathbf{N_f}(\mathbf{1})$ and quit
18:    **else**
19:      $\mathbf{N_f} \leftarrow \hat{\mathbf{N}}_{\mathbf{f}}$
20:    **end if**
21: **end if**

B. Branch and Bound Algorithm

The solution obtained with the ARASP algorithm belongs to $\mathbb{R}$ which is not suitable for practical systems. Indeed, the processing gain N_f for each user needs to be an integer belonging to $\{1, \cdots, N_c\}$ as stated by the constraint (c_4) in (18). The well-known branch and bound (BB) algorithm is particularly adapted to this kind of problem, i.e. integer programming problem [17]. The BB algorithm is not a heuristic in the sense that it provides a provable upper and lower bound of the optimal solution [15]. However, we are still dealing with non linearities in our problem and we are hence facing up to the same issue than the one exposed above which prevents to find the global optimal solution surely. However, the BB algorithm remains a benchmark to evaluate other heuristics.

The BB principle is firstly to solve the signomial programming problem in (22) for which the integer constraint in (18) has been released leading to the solution $\mathbf{N_f^*} \in \mathbb{R}$. For a given non-integer entry in the vector $\mathbf{N_f^*}$, let say $N_f^{(j)}$,

two subproblems are created, i.e. P_1 and P_2, the former with the additional constraint $N_f^{(j)} \leq \left\lfloor N_f^{(j)*} \right\rfloor$ and the latter with $N_f^{(j)} \geq \left\lceil N_f^{(j)*} \right\rceil$ [3]. This operation is repeated until the vector $\mathbf{N_f}$ only contains integer entries. We propose the adaptive rate allocation with branch and bound and signomial programming (ARABBSP) stated in the algorithm 2.

Algorithm 2 Adaptive Rate Allocation with Branch and Bound and Signomial Programming (ARABBSP)

Require: $N_u \geq 1$, $N_c > 1$,
Ensure: $\mathbf{N_f} \in \mathbb{N}$ allocation for feasible problems
 1: find a feasible $\hat{\mathbf{N}}_{\mathbf{f}}$
 2: create $P = \left\{\min \prod_u N_f^{(u)}, \text{s.t. constraints in (22)}\right\}$
 3: **while** $P \neq \emptyset$ **do**
 4:    solve all problems in P with ARASP
 5:    remove all infeasible problems from P
 6:    **if** all solutions $\in \mathbb{N}$ **then**
 7:      Choose the one minimizing $\prod_u N_f^{(u)}$
 8:    **else if** for a given $P_i \in P$, at least one $N_f^{(j)}$ is non-integer **then**
 9:      remove P_i from P
10:      create the new problem P_i with the constraints in P plus $N_f^{(j)} \leq \left\lfloor N_f^{(j)*} \right\rfloor$
11:      create the new problem P_{i+1} with the constraints in P plus $N_f^{(j)} \geq \left\lceil N_f^{(j)*} \right\rceil$
12:    **end if**
13: **end while**

The first step in the algorithm 2 (or in the ARASP algorithm) is very important and the question how to find a feasible vector *in an efficient way* is not trivial. A feasible vector is typically an $\mathbf{N_f}$ satisfying the problem constraints, i.e. the SINR constraints essentially. We start by allocating the same number of frames to each user starting with $N_f = 1$ and incrementing N_f up to N_c until all the SINR constraints are satisfied. If some users do not fulfill their QoS constraints at the end of this procedure, a random search on $\mathbf{N_f}$ is performed until the constraints are satisfied or a maximum number of iterations is achieved. In this case, the search is stopped and a user is removed from the resource allocation controller and the procedure reboots. We draw the reader's attention that there is no general method to find a feasible point due to the mixed signomial and integer nature of the problem[4].

C. Adaptive rate allocation heuristic

Branch and bound procedures give good results in general but remain often relatively complex. The complexity may grow exponentially with the problem dimensions in some cases [15] which can motivate the search for practical algorithms with lower complexity even though their good performances cannot be proved formally. The BB algorithm proposed above can be

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[3]It is worth noting that $N_f^{(j)*}$ violates these new constraints
[4]Finding a feasible solution for very difficult problems (such as NP problems) can be an issue

complex and it might be worth searching for a lower complexity processing gain allocation procedure. Let us focus on the optimization problem (18). One way to formulate the problem is to find the minimum $N_f^{(u)}$ for each user such as they satisfy their SINR constraint, i.e. (c_1) in (18). The constraint (c_1) in (18) can be rewritten in order to be a third degree equation in $N_f^{(u)}, \forall u \in \{1, \cdots, N_u\}$ as expressed in [1, eq. (18)] and reported in eq. (23) in the paper for the sake of readability. By choosing the minimum $N_f^{(u)}$ solving this equation for each user, we propose the following heuristic stated in Algorithm 3 for the processing gain allocation procedure.

Algorithm 3 Adaptive Rate Allocation Algorithm (ARAA)

Require: $N_u \geq 1$, $N_c > 1$,

Ensure: Assign valid number of frames to users.

1: Initialize M, $I = \{1, \cdots, N_u\}$, $\mathbf{N_f} = [1, \cdots, 1]$. Sort I such that $P_1 > \cdots > P_{N_u}$
2: $N_u' \leftarrow N_u$, $i = 0$
3: **while** $\text{SINR}_u < \Gamma_{\min} \ \forall u \in I$ **do**
4: $\quad u = N_u'$, $i = i + 1$
5: $\quad$ **while** $u \geq 1$ **do**
6: $\quad\quad N_f^{(u)} \leftarrow$ solve $F\left(N_f^{(u)}, N_f^{(u')}\right)_{u' \neq u} \geq 0$
7: $\quad\quad$ **if** $N_f^{(u)} > N_c$ or $N_f^{(u)} < 1$ **then**
8: $\quad\quad\quad N_u' \leftarrow N_u' - 1$ and remove u from I
9: $\quad\quad\quad u = N_u'$ and $\forall u \in I, N_f^{(u)} = 1$
10: $\quad\quad$ **else if** $i > M$ **then**
11: $\quad\quad\quad N_u' \leftarrow N_u' - 1$ and remove N_u' from I
12: $\quad\quad\quad u = N_u'$ and $\forall u \in I, N_f^{(u)} = 1$, $i = 0$
13: $\quad\quad$ **else**
14: $\quad\quad\quad u = u - 1$
15: $\quad\quad$ **end if**
16: $\quad$ **end while**
17: $\quad$ Update $\text{SINR}_u \ \forall u \in I$
18: **end while**

The algorithm starts by allocating the minimum N_f to each user, i.e. $N_f^{(u)} = 1, \forall u \in I$ and by sorting the users in decreasing order according to the received power at the BS. The SINR of each user is then computed. If some users do not fulfill their SINR constraint, the number of frames of each user is updated starting by the farthest user from the BS because it experiences the lowest SINR *a priori*. The minimum N_f for this user is computed by solving the inequality in step 6 of the algorithm ARAA[5]. If a valid solution is found, the algorithm goes to the next user and so on. If no solution is found, the user is removed from the resource allocation controller and the algorithm reboots. Once all the users' processing gains have been updated, the SINR of each user is checked, if all SINR constraints are fulfilled the algorithm stops. If not, the last user is removed from the network and the algorithm restarts. This algorithm has a linear complexity with the number of users and hence the N_f allocation is very simple.

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[5]The expression of $F\left(N_f^{(u)}, N_f^{(u')}\right)_{u' \neq u}$ can be found in [1, eq. (18)] and has been reported in eq. (23) at the top of the next page.
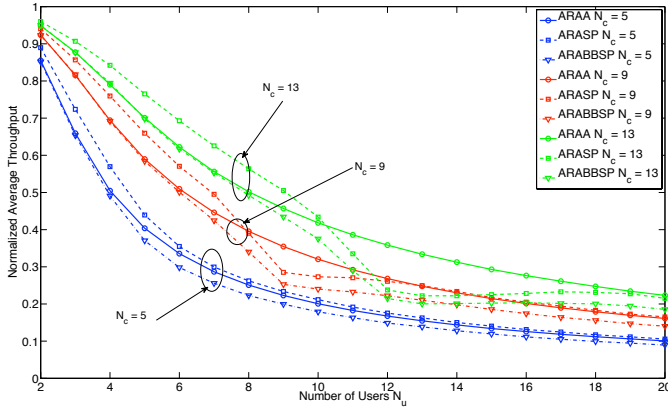
V. NUMERICAL RESULTS AND DISCUSSION

In this section, we investigate the relative performances of the three algorithms proposed in this paper i.e. ARASP, ARABBSP and ARAA. The performances are investigated in terms of the global throughput and the average starvation rate, i.e. the average rate of users without resources. The channel and system parameters used in the simulations are summarized in Table I. The cell is assumed to be a square of side-length normalized to the unity and the base station is assumed to occupy the center of the cell. The cartesian coordinates of each user, i.e. x_u and y_u, are randomly selected from a uniform distribution in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. We assume that the path loss exponent is 2 as reminded in Table I (under the notation PL) and hence the received power after the path loss propagation is proportional to $1/\left(x_u^2 + y_u^2\right)$.

In Fig. 4, the normalized average throughput and the average starved user rate for the ARASP, ARABBSP and ARAA algorithms are investigated w.r.t. the number of users N_u labeled on the number of chips N_c. The normalized throughput is the global throughput of the cell, in Mbps, normalized w.r.t. the throughput which would be achieved if all users would transmit at their maximum data rate without any QoS constraint. While the average starved user rates are of the same order of magnitude between the different algorithms for a same number of chips, as it can be inferred from Fig. 4(b), there is an interesting behavior of the throughput of the ARASP and ARABBSP revealed by Fig. 4(a). We could first think that since the ARASP solves the allocation problem in $\mathbb{R}$, the average throughput obtained would be greater than the throughput of the ARAA. However, if the number of users and the number of chips are of the same order of magnitude, e.g. $N_c = 9$ and $N_u = 9$, then the throughput of the ARASP falls below the throughput of the ARAA and increases again for an increasing number of users. This non expected and interesting behavior can be explained by the fact that the average N_f value per user is greater for the ARASP than for the ARAA, as illustrated in Fig. 5 and hence leads to a lower average throughput. We remind that the ARASP algorithm (and hence ARABBSP) needs to be initialized by a feasible $\mathbf{N_f}$ (cf. Section IV). This vector is generally of the form $\mathbf{N_f} = q \cdot \mathbf{1}$, where $q \in \{1, \cdots, N_c\}$ and $\mathbf{1}$ is a vector with all entries are equal to one. It leads to a higher average number of frames per user compared to the ARAA algorithm which starts with $\mathbf{N_f} = \mathbf{1}$ and computes the minimal N_f for each user. Moreover, the problem can start from a feasible point but can become infeasible as the relative values of $N_f^{(u)}$ change between the users. This behavior is due to the non linearities involved in the problem (22) as previously discussed in Section IV. Moreover, let us remind that the original problem is far from convex and the ARASP is based on a local convex approximation of the problem. Hence, there is no reason that
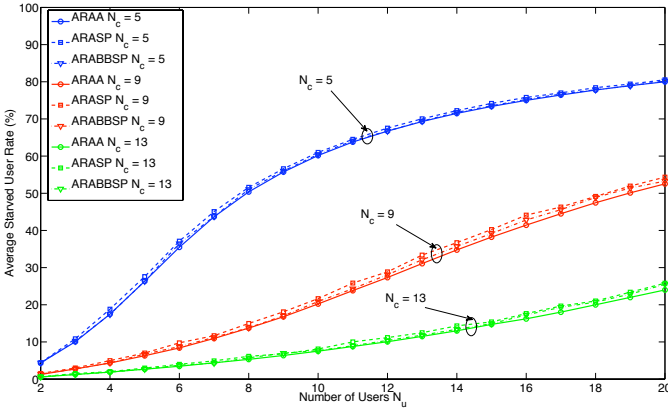
Parameters	λ	γ	N_p	L_r	T_c	$\Gamma_{\min}$	PL
Values	2.1 ns^{-1}	12 ns	25	3	5 ns	10 dB	2

TABLE I
SYSTEM PARAMETERS USED IN THE SIMULATIONS

$$F\left(N_f^{(u)}, N_f^{(u')}\right)_{u' \neq u} = N_c^2 \Lambda \Gamma_{\min} \sum_{v \in I_-} \frac{P_v}{N_f^{(v)}} N_f^{(u)3} + \left(P_u G_u - 3N_c^2 \Lambda \Gamma_{\min} \sum_{v \in I_-} P_v\right) N_f^{(u)2} - \Gamma_{\min} \times$$

$$\left\{ \Lambda \left[\sum_{v \in I_-} \left(3N_c^3 - 3N_c + \frac{1}{N_f^{(v)}}\right) P_v + 3N_c \sum_{\substack{u' \in I_+ \\ u' \neq u}} \left(N_c N_f^{(u')} + N_c^2 - 1\right) P_{u'} \right] \right.$$

$$\left. + \frac{N_0 V_n}{2} \right\} N_f^{(u)} + \Lambda \Gamma_{\min} \sum_{\substack{u' \in I_+ \\ u' \neq u}} \left(N_c^2 N_f^{(u')2} - 1\right) P_{u'} \qquad (23)$$

(a) Normalized Average Throughput

(b) Average Starved User Rate

Fig. 4. Normalized Average Throughput and Average Starvation Rate w.r.t. the number of users N_u and labeled on N_c

Fig. 5. Average N_f allocated per user w.r.t. the number of users and labeled on N_c

(i.e. ARABBSP) does not perform better than the ARAA. We can observe the same kind of behaviors in Fig. 6 where the average normalized throughput, Fig. 6(a), and the starvation rate, Fig. 6(b), have been plotted w.r.t. the number of chips N_c and labeled on the number of users N_u. The throughput of the ARASP suddenly falls below the throughput of the ARAA when N_u and N_c are of the same order of magnitude, excepted for $N_u = 5$ for which there is no crossover point; the ARASP outperforms the ARAA which outperforms the ARABBSP. We also draw the reader's attention that the algorithms presented above can be applied for heterogeneous QoS requirements between users, i.e. when different SINR thresholds among users are considered, and all the materials developed in this paper remain valid in the aforementioned case.

Moreover, $\Gamma_{\min}$ has an impact on the network performance. For instance, a lower $\Gamma_{\min}$ would imply that more users in average would be satisfied and the user starvation rate would be lower. An opposite conclusion would arise for an higher $\Gamma_{\min}$ requirement. However, $\Gamma_{\min}$ is not a parameter which can be optimized; it is a constraint of the system, it is a QoS requirement for each user or a set of users. The system has to perform the resource allocation procedure in order to optimize the objective function while satisfying in the same time the SINR requirement for all users. If the problem is too constrained, all the users cannot be satisfied and some of them are in starvation.

The ARASP is an iterative procedure calling a geometric

a good heuristic cannot be better than the ARASP.

The behavior of the ARABBSP is similar to the ARASP since the former is derived from the later. However the average throughput of the ARABBSP is lower than the throughput of the ARASP since the set of solutions belongs to $\mathbb{N}$ for the former instead of $\mathbb{R}$ for the later. For $N_c = 5$, the throughput of the ARASP is slightly above the one of the ARAA while the throughput of the ARABBSP is slightly below the ARAA. For higher number of chips, e.g. $N_c = 9$ or $N_c = 13$, the ARASP outperforms the ARAA when $N_u < N_c$ but the behavior is inverted for $N_u \geq N_c$. But in any cases, the BB algorithm
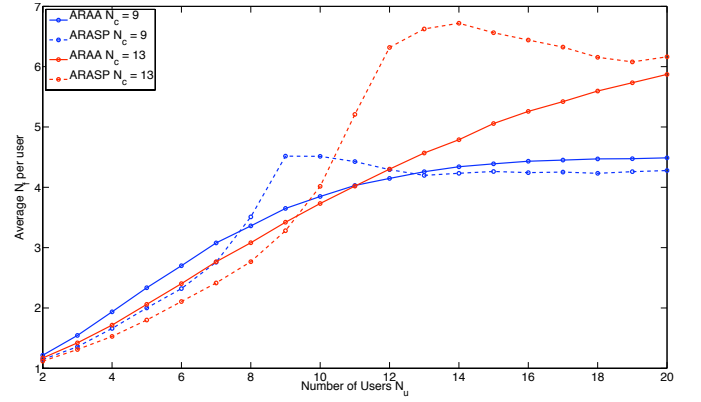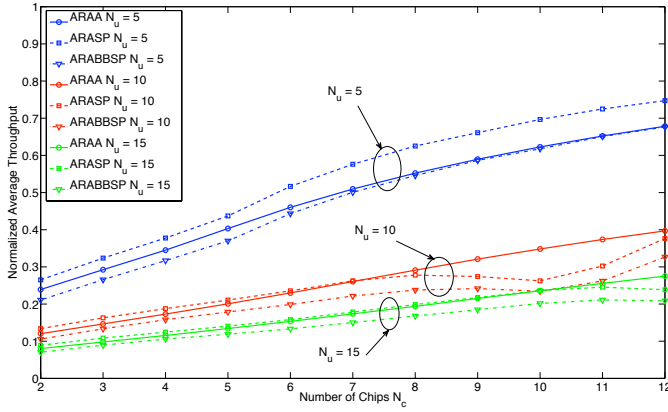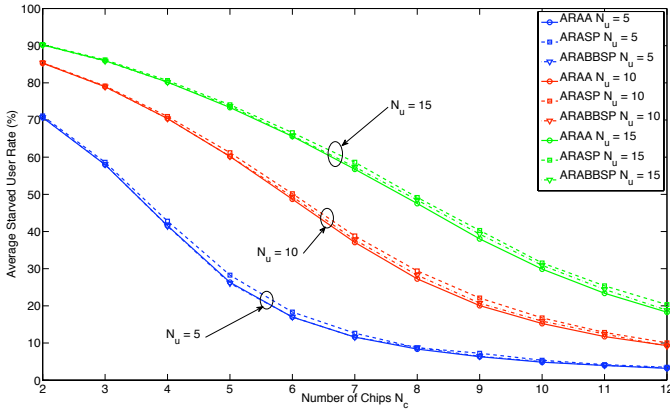
(a) Normalized Average Throughput

(b) Average Starved User Rate

Fig. 6. Normalized Average Throughput and Average Starvation Rate w.r.t. the number of chips N_c and labeled on N_u

programming solver at each iteration. The resolution of geometric programming problems is now quite efficient with interior point-based methods and can be solved in a polynomial time with the problem size, i.e. the number of users, in the worst case. The number of iterations depends on the precision required and can be fixed off-line. The ARABBSP is based on the ARASP since the latter is called in the former (cf. step 4 in Algorithm 2). Moreover, the branch and bound procedure itself has a higher complexity which can grow exponentially, in the worst case, with the number of users. However, it may converge quickly if the initial guess is good. On the other hand, the ARAA has a linear complexity with the number of users and hence is very simple.

From a more practical point of view, the multi-rate allocation is performed at the base station. The resource allocation controller needs to evaluate the average SINR of each user which implies to know the average received power i.e. P_u $\forall u$, the current number of frames for each user i.e. $N_f^{(u)}$ $\forall u$, the SINR requirement for each user i.e. $\Gamma_{\min}$ and the basic channel parameters obtained in the channel estimation procedure i.e. λ, γ and N_p. All these parameters can be obtained during the first channel estimation and signal acquisition procedure provided that the time selectivity of the channel is slow enough to allow the resource allocation procedure to be performed and the results to be sent back to mobile stations. Other parameters

are obviously known at the base station since they are full-part of the system, e.g. N_c, L_r. A feedback channel is also needed between the base station and the mobile stations in order to communicate the number of frames $N_f^{(u)}$ to the user u.

VI. CONCLUSIONS

In this paper, multi-rate resource allocation for TH-UWB wireless communications have been investigated. We have first provided a closed-form expression for the average multiple access interference for TH-UWB multi-rate systems. The closed form expression has been derived by extending the intercode cross-correlation model to the multi-rate context making the multiple access interference model more general than those provided in the existing literature. The multi-rate resource allocation problem has been revisited by expressing the processing gain allocation issue via a general signomial programming problem. We have proved that the signomial problem can be locally approximated by a geometric programming problem which can be solved efficiently. From this, an adaptive rate allocation procedure with signomial programming has been provided as well as a branch and band based algorithm allowing to find processing gains as integers. We finally proposed a very simple adaptive rate allocation procedure with linear complexity. The performances of these new resource allocation algorithms have been compared according to their maximum throughput and average starvation rate. The investigations have shown that the ARAA algorithm outperforms the branch and band algorithm in both average throughput and average starvation rate with a lower computational complexity.

VII. APPENDIX

A. Proof of Lemma 1

The autocorrelation function r_{ww} in (8) is non zero if and only if:

$$-T_{r_{ww}} \leq \left(Q_{u'}^{n,l} + i\right) T_s^{(u')} + \left(q_{u'}^{n,l} + j - j_u\right) T_c + \epsilon_{u'}^{n,l} \leq T_{r_{ww}} \tag{24}$$

with $T_{r_{ww}}$ denotes the support of the function r_{ww}. Moreover, $0 \leq \epsilon_{u'}^{n,l} < T_c$ hence (24) can be changed in:

$$-T_{r_{ww}} - T_c < \left(Q_{u'}^{n,l} + i\right) T_s^{(u')} + \left(q_{u'}^{n,l} + j - j_u\right) T_c \leq T_{r_{ww}} \tag{25}$$

Since $T_{r_{ww}} < T_c$, from (25) we can assess:

$$-2T_c < \left(Q_{u'}^{n,l} + i\right) T_s^{(u')} + \left(q_{u'}^{n,l} + j - j_u\right) T_c < T_c \tag{26}$$

Let us consider the two cases i) $\alpha^{(u')} < 1$ and ii) $\alpha^{(u')} \geq 1$. Let us start with i) $\alpha^{(u')} < 1$. We have $\left\lceil \alpha^{(u')} \right\rceil - 1 = 0$. Hence, knowing that $-2 < Q_{u'}^{n,l} + i \leq \left\lceil \alpha^{(u')} \right\rceil - 1$, it follows that 1) $i = -1 - Q_{u'}^{n,l}$ or 2) $i = -Q_{u'}^{n,l}$. Let us continue with the first case for which (26) becomes $-2T_c < \left(q_{u'}^{n,l} + j - j_u - N_c N_f^{(u')}\right) T_c < T_c$. From this, it follows that:

$$q_{u'}^{n,l} + j - j_u - N_c N_f^{(u')} = \begin{cases} 0 \\ -1 \end{cases} \tag{27}$$

Hence, according to these latter cases, (8) can be written as:

$$
\begin{aligned}
y_{u',u}^{n,l}(\theta_{u'}) \;=\; & d_{u'}\left(-Q_{u'}^{n,l}-1\right) \\
& \times \left[\sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u)r_{ww}\left(\epsilon_{u'}^{n,l}\right) \right. \\
& \left. + \sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u)\times \right. \\
& \left. \qquad\qquad r_{ww}\left(\epsilon_{u'}^{n,l}-T_c\right) \right]. \quad (28)
\end{aligned}
$$

Let us consider the first case in (27). Then we have $j = j_u + N_cN_f^{(u')} - q_{u'}^{n,l}$. Considering $0 \le j_u \le N_cN_f^{(u)}-1$, the following inequalities hold:

$$
N_cN_f^{(u')}-q_{u'}^{n,l} \le j \le \left(\alpha^{(u')}+1\right)N_cN_f^{(u')}-1-q_{u'}^{n,l}. \quad (29)
$$

Moreover $j \le N_cN_f^{(u')}-1$, hence $N_cN_f^{(u')}-q_{u'}^{n,l} \le j \le N_cN_f^{(u')}-1$. We also have $j_u = j - N_cN_f^{(u')} + q_{u'}^{n,l}$ and hence $j_u \in \left\{0,\cdots,q_{u'}^{n,l}-1\right\}$. It comes:

$$
\sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u) = \sum_{j_u} c_u(j_u) \times \\
c_{u'}\left(j_u - q_{u'}^{n,l} + N_cN_f^{(u')}\right) \quad (30)
$$

and $c_{u'}$ is $N_cN_f^{(u')}$ periodic and $c_u(j_u) = 0$ if $j_u > N_cN_f^{(u)}-1$. We finally have:

$$
\sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u) = \\
\sum_{j_u=0}^{\min\left(q_{u'}^{n,l},N_cN_f^{(u)}\right)-1} c_u(j_u)\,c_{u'}\left(j_u - q_{u'}^{n,l}\right) := C_{u,u'}^{-1}\left(q_{u'}^{n,l}\right). \quad (31)
$$

When the second case in (27) is considered, the same kind of results are derived leading to $C_{u,u'}^{-1}\left(q_{u'}^{n,l}+1\right)$. Considering the second case above named 2), i.e. $Q_{u'}^{n,l}+i = 0$, and similar steps than above, leads to the definition of $C_{u,u'}^0$ in eq. (11) which closes the case i) $\alpha^{(u')} < 1$.

Let us now describe briefly the steps for ii) $\alpha^{(u')} \ge 1$. In this case, $Q_{u'}^{n,l}+i = k$, $k \in \left\{-1,\cdots,\left\lceil\alpha^{(u')}\right\rceil-1\right\}$. The inequalities (26) become $-2T_c < \left(q_{u'}^{n,l}+j-j_u+kN_cN_f^{(u')}\right)T_c < T_c$. From this, it follows that:

$$
q_{u'}^{n,l}+j-j_u+kN_cN_f^{(u')} = \begin{cases} 0 \\ -1 \end{cases} \quad (32)
$$

Eq. (8) can now be written as:

$$
\begin{aligned}
y_{u',u}^{n,l}(\theta_{u'}) = & \sum_{k=-1}^{\left\lceil\alpha^{(u')}\right\rceil-1} d_{u'}\left(k-Q_{u'}^{n,l}\right) \times \\
& \left[\sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u)r_{ww}\left(\epsilon_{u'}^{n,l}\right) + \right. \\
& \left. \sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u)r_{ww}\left(\epsilon_{u'}^{n,l}-T_c\right) \right]. \quad (33)
\end{aligned}
$$

Let us consider the first case in (32). Then we have $j = j_u - kN_cN_f^{(u')}-q_{u'}^{n,l}$. Considering $0 \le j_u \le N_cN_f^{(u)}-1$, the following inequalities hold:

$$
-kN_cN_f^{(u')}-q_{u'}^{n,l} \le j \le N_cN_f^{(u)}-kN_cN_f^{(u')}-q_{u'}^{n,l}-1. \quad (34)
$$

The case $Q_{u'}^{n,l}+i = -1$ has already been discussed above and in the following only the cases $k \ge 0$ will be considered. Moreover, $j \ge 0$ hence $0 \le j \le N_cN_f^{(u)}-kN_cN_f^{(u')}-q_{u'}^{n,l}-1$. We also have $j_u = j + kN_cN_f^{(u')}+q_{u'}^{n,l}$ and hence $j_u \in \left\{q_{u'}^{n,l}+kN_cN_f^{(u')},\cdots,N_cN_f^{(u)}-1\right\}$. The symbol $d_{u'}\left(k-Q_{u'}^{n,l}\right)$ is related to the intercode interference:

$$
\sum_{j=0}^{N_cN_f^{(u')}-1} \sum_{j_u=0}^{N_cN_f^{(u)}-1} c_{u'}(j)c_u(j_u) = \sum_{j_u} c_u(j_u) \times \\
c_{u'}\left(j_u - q_{u'}^{n,l} - kN_cN_f^{(u')}\right). \quad (35)
$$

If $j_u \ge q_{u'}^{n,l}+(k+1)N_cN_f^{(u')}$, the intercode interference term is related to the symbol $d_{u'}\left(k+1-Q_{u'}^{n,l}\right)$. Due to the definition range of $q_{u'}^{n,l}$, if $q_{u'}^{n,l}+(k+1)N_cN_f^{(u')} \le N_cN_f^{(u)}-1$, it implies $0 \le k \le \alpha^{(u')}-2$. For the sake of brevity, let us consider the general case $\alpha^{(u')} \in \mathbb{Q}^*\backslash\mathbb{N}^{*6}$, we hence have $0 \le k \le \left\lceil\alpha^{(u')}\right\rceil-3$ and:

$$
C_{u,u'}^k\left(q_{u'}^{n,l}\right) := \sum_{j_u=q_{u'}^{n,l}+kN_cN_f^{(u')}}^{q_{u'}^{n,l}+(k+1)N_cN_f^{(u')}-1} c_u(j_u) \times \\
c_{u'}\left(j_u - q_{u'}^{n,l} - kN_cN_f^{(u')}\right). \quad (36)
$$

For $k = \left\lceil\alpha^{(u')}\right\rceil-2$, if $q_{u'}^{n,l} \le \left(\alpha^{(u')}-\left\lfloor\alpha^{(u')}\right\rfloor\right)N_cN_f^{(u')}-1$, then (36) is used, else the upper bound of the summation is $N_cN_f^{(u)}-1$. It follows that:

$$
C_{u,u'}^k\left(q_{u'}^{n,l}\right) := \sum_{j_u=q_{u'}^{n,l}+kN_cN_f^{(u')}}^{\min\left(q_{u'}^{n,l}+(k+1)N_cN_f^{(u')},N_cN_f^{(u)}\right)-1} c_u(j_u) \times \\
c_{u'}\left(j_u - q_{u'}^{n,l} - kN_cN_f^{(u')}\right). \quad (37)
$$

---

[6]Similar results can be derived for $\alpha^{(u')} \in \mathbb{N}^*$. The notation $\mathbb{S}^*$, $\mathbb{S}$ being a set, stands for $\mathbb{S}\backslash\{0\}$

$$\mathbb{E}_c\left[C_{u,u'}^{-1}(q)^2\right] = \frac{\min\left(q, N_c N_f^{(u)}\right)}{N_c^4}\left(\min\left(q, N_c N_f^{(u)}\right) + N_c^2 - 1\right), \tag{38}$$

if $\alpha^{(u')} < 1$:

$$\mathbb{E}_c\left[C_{u,u'}^0(q)^2\right] = \frac{N_c N_f^{(u)} - \min\left(q, N_c N_f^{(u)}\right)}{N_c^4}\left(N_c N_f^{(u)} - \min\left(q, N_c N_f^{(u)}\right) + N_c^2 - 1\right), \tag{39}$$

if $\alpha^{(u')} \geq 1$:

$$\mathbb{E}_c\left[C_{u,u'}^k(q)^2\right] = \begin{cases} \frac{N_f^{(u')}}{N_c^3}\left(N_c N_f^{(u')} + N_c^2 - 1\right),\ 0 \leq k \leq \left\lceil\alpha^{(u')}\right\rceil - 3,\ \forall q, \\[2mm] \frac{N_f^{(u')}}{N_c^3}\left(N_c N_f^{(u')} + N_c^2 - 1\right), k = \left\lceil\alpha^{(u')}\right\rceil - 2,\ \text{and}\ q \leq \left(\alpha^{(u')} - \left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')} - 1, \\[2mm] \frac{\left(1+\alpha^{(u')}-\left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')}-q}{N_c^4}\left(\left(1+\alpha^{(u')}-\left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')} - q + N_c^2 - 1\right), k = \left\lceil\alpha^{(u')}\right\rceil - 2, \\[1mm] \hspace{6cm} \text{and}\ q \geq \left(\alpha^{(u')} - \left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')}, \\[2mm] \frac{\left(\alpha^{(u')}-\left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')}-q}{N_c^4}\left(\left(\alpha^{(u')} - \left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')} - q + N_c^2 - 1\right), k = \left\lceil\alpha^{(u')}\right\rceil - 1, \\[1mm] \hspace{6cm} \text{and}\ q \leq \left(\alpha^{(u')} - \left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')} - 1, \\[2mm] 0, k = \left\lceil\alpha^{(u')}\right\rceil - 1,\ \text{and}\ q \geq \left(\alpha^{(u')} - \left\lfloor\alpha^{(u')}\right\rfloor\right)N_c N_f^{(u')} \end{cases} \tag{40}$$

If $k = \left\lceil\alpha^{(u')}\right\rceil - 1$ then $(k+1)N_c N_f^{(u')} > N_c N_f^{(u)}$. Hence, the last intercorrelation term is given by:

$$C_{u,u'}^k\left(q_{u'}^{n,l}\right) := \sum_{j_u = q_{u'}^{n,l} + k N_c N_f^{(u')}}^{N_c N_f^{(u)}-1} c_u(j_u) \times$$
$$c_{u'}\left(j_u - q_{u'}^{n,l} - k N_c N_f^{(u')}\right) \tag{41}$$

and the eq. (12) is proved and the proof is complete.

B. Sketch of proof of Theorem 1

The vector $c_u\ \forall u$, is the realization of an i.i.d. random vector whose each component is a Bernoulli random variable with parameter $p = 1/N_c$ [14]. Hence, $C_{u,u'}^k(q)$ is a binomial random variable depending on k and on $\alpha^{(u')}$. For the reader's convenience, the second order moments of $C_{u,u'}^k(q)$, depending on k, are expressed in (38), (39) and (40) on top of the page. After summation on k and q, and tedious algebraic manipulations, the expressions in Theorem 1 are obtained and the proof is complete.

C. Proof of Proposition 1

Expanding the constraint (c_1) in (18) into a sum of monomials of the form $N_f^{(u')\alpha_1} N_f^{(u)\alpha_2}$, with $u,\ u' \in I$ and $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, (c_1) can be re-arranged as:

$$p_u(\mathbf{N_f}) - q_u(\mathbf{N_f}) \leq 1,\ \forall u \in I, \tag{42}$$

where p_u and q_u being two posynomials in a standard form [15], $p_u(\mathbf{N_f})$ is expressed as in eq. (20) and q_u is:

$$q_u(\mathbf{N_f}) = N_c^2 \Gamma_{\min}(u)\Lambda \sum_{u' \in I_+} P_{u'} N_f^{(u')2} N_f^{(u)-2} +$$
$$N_c^2 \Gamma_{\min}(u)\Lambda \sum_{v \in I_-} P_v N_f^{(v)-1} N_f^{(u)}. \tag{43}$$

By simply moving the posynomial q_u on the right side of the inequality (42), (c_1) can be expressed as $p_u(\mathbf{N_f}) \leq r_u(\mathbf{N_f})$ with r_u is given as in Proposition 1.

The problem in this form remains signomial. Let us consider the best monomial approximation $f_u\left(\hat{\mathbf{N}}_{\mathbf{f}}\right) = \beta_u \prod_{j=1}^{N_u}\left(\hat{N}_f^{(j)}\right)^{\alpha_u}$ of the posynomial $r_u\left(\hat{\mathbf{N}}_{\mathbf{f}}\right)$ [15]. Around the point $\hat{\mathbf{N}}_{\mathbf{f}}$, we have:

$$\begin{cases} f_u\left(\hat{\mathbf{N}}_{\mathbf{f}}\right) & = & r_u\left(\hat{\mathbf{N}}_{\mathbf{f}}\right) & \forall u \in I \\ \nabla_{N_f^{(u)}} f_u\left(\hat{\mathbf{N}}_{\mathbf{f}}\right) & = & \nabla_{N_f^{(u)}} r_u\left(\hat{\mathbf{N}}_{\mathbf{f}}\right) & \forall u \in I \end{cases} \tag{44}$$

From the first equality, we can express β_u as in eq. (19). Taking the partial derivative in the second equality we get $\forall u \in I$:

$$\alpha_u \beta_u N_f^{(u)-1} \prod_{j=1}^{N_u}\left(N_f^{(j)}\right)^{\alpha_u} = N_c^2 \Gamma_{\min}(u)\Lambda \times$$
$$\left[\sum_{v \in I_-} P_c N_f^{(v)-1} - 2\sum_{u' \in I_+} P_{u'} N_f^{(u')2} N_f^{(u)-3}\right]. \tag{45}$$

Substituting the expression of β_u in eq. (19) in (45), the expression of α_u in eq. (19) is obtained and the proof is complete.

REFERENCES

[1] P. Mary, I. Fijalkow, and C. Poulliat, "Adaptive Rate Allocation Scheme for Uplink TH-UWB Networks," in *IEEE International Conference on Communications (ICC)*, June 2011.

[2] "IEEE Standard for Information Technology - Telecommunications and Information Exchange Between Systems - Local and Metropolitan Area Networks - Specific Requirement Part 15.4: Wireless Medium Access Control (MAC) and Physical Layer (PHY) Specifications for Low-Rate Wireless Personal Area Networks (WPANs)," pp. 1 –203, 2007.

[3] M. Z. Win and R. A. Scholtz, "Ultra-Wide Bandwidth Time-Hopping Spread-Spectrum Impulse Radio for Wireless Multiple-Access Communications," *IEEE Transactions on Communications*, vol. 48, no. 4, pp. 679–689, Apr. 2000.

[4] C. J. Le Martret and G. B. Giannakis, "All-Digital Impulse Radio with Multiuser Detection for Wireless Cellular Systems," *IEEE Transactions on Communications*, vol. 50, no. 9, pp. 1440–1450, Sep. 2002.

[5] C. J. Le Martret, A.-L. Deleuze, and P. Ciblat, "Optimal Time-Hopping Codes for Multi-User Interference Mitigation in Ultra-Wide Bandwidth Impulse Radio," *IEEE Transactions on Wireless Communications*, vol. 5, no. 6, pp. 1516 –1525, June 2006.

[6] B. Radunovic and J. Y. Le Boudec, "Optimal power control, scheduling, and routing in UWB networks," *IEEE Journal on Selected Areas in Communications*, vol. 22, no. 7, pp. 1252–1270, Sep. 2004.

[7] S. A. Jafar and A. Goldsmith, "Adaptive Multirate CDMA for Uplink Throughput Maximization," *IEEE Transactions on Wireless Communications*, vol. 2, no. 3, pp. 218–228, Mar. 2003.

[8] Y. Guo and B. Aazhang, "Capacity of Multi-Class Traffic CDMA System with Multiuser Receiver ," in *IEEE Wireless Communications and Networking Conference (WCNC)*, 1999, pp. 500–504.

[9] H. Wymeersch, G. Zussman, and M. Z. Win, "SNR Analysis for MultiRate UWB-IR," *IEEE Communications Letters*, vol. 11, pp. 49–51, 2007.

[10] H. Wymeersch and M. Moeneclaey, "ML Rate Detection for Multi-rate TH-UWB Impulse Radio," in *IEEE International Conference on Ultra-Wideband (ICU)*, sept. 2005, pp. 391 – 395.

[11] M. Nasiri-Kenari and M. G. Shayesteh, "Performance Analysis and Comparison of Different Multirate TH-UWB Systems: Uncoded and Coded Schemes," *IEE Proceedings - Communications*, vol. 152, no. 6, pp. 833–844, 2005.

[12] F. Kharrat-Kammoun, C. J. Le Martret, and P. Ciblat, "Performance analysis of ir-uwb in a multi-user environment," *IEEE Transactions on Wireless Communications*, vol. 8, no. 11, pp. 5552 –5563, nov. 2009.

[13] A. F. Molisch, J. R. Foerster, and M. Pendergrass, "Channel models for ultrawideband personal area networks," *IEEE Wireless Communications*, vol. 10, no. 6, pp. 14–21, Dec. 2003.

[14] A.-L. Deleuze, C. J. Le Martret, P. Ciblat, and E. Serpedin, "Cramer-Rao Bound for Channel Parameters in Ultra-Wide Band Based System," in *IEEE 5th Workshop on Signal Processing Advances in Wireless Communications (SPAWC)*, july 2004.

[15] S. Boyd, S.-J. Kim, L. Vandenberghe, and A. Hassibi, "A tutorial on geometric programming," *Optimization and Engineering*, vol. 8, no. 1, pp. 67–127, 2007.

[16] S. Boyd and L. Vandenberghe, *Convex Optimization*. United Kingdom: Cambridge University Press, 2004.

[17] R. E. Moore, "Global optimization to prescribed accuracy," *Computers & Mathematics with Applications*, vol. 21, no. 6-7, pp. 25–39, 1991.

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